DOCUMENT RESUME

ED 348 071 JC 880 364

AUTHOR	Cohen, Don, Ed.
TITLE	The AMATYC Review, Fall 1987, Spring 1988.
INSTITUTION	American Mathematical Association of Two-Year Colleges.
REPORT NO	ISSN-0740-8404
PUB DATE	88
NOTE	142p.
AVAILABLE FROM	AMATYC, Mott Community College, 1401 E. Court St., Flint, MI 48502 (3 issues free with $25 membership).
PUB TYPE	Collected Works - Serials (022)
JOURNAL CIT	The AMATYC Review; v9 n1-2 Fall 87 Spr 88
EDRS PRICE	MF01 Plus Postage. PC Not Available from EDRS.
DESCRIPTORS	*College Mathematics; Community Colleges; Content Area Writing; Functions (Mathematics); *Mathematical Concepts; *Mathematics Instruction; Teaching Methods; Trigonometry; Two Year Colleges

ABSTRACT

Designed as an avenue of communication for mathematics educators concerned with the views, ideas, and experiences of two-year college students and teachers, this journal contains articles on mathematics exposition and education, and regular features that present book and software reviews and math problems. The first of two issues of volume 9 contains the following major articles: (1) "The Role of the Computer in Mathematical Reasoning," by Bill Leonard and Harris Schultz; (2) "The Trigonometric Functions and Arcs," by Richard F. Maruszewski; (3) "Apparent Aging: An Explanation of a Common Paradox," by Joseph Browne; (4) "Hindu Romance with Quadratic Functions," by Gurcharan Singh Bhalla; (5) "A Unified Approach to the Circular and Hyperbolic Functions," by Mary Coughlin; (6) "QUESTIONBOOKS: Using Writing To Learn in Mathematics," by Agnes Azzolino and Robert G. Roth; (7) "What Differential Equations Books Lack," by George E. Abdo; and (8) "Future Shock: Hand Held Computers," by Gregory D. Foley. The second issue of volume 9 contains the following major articles: (1) "The Clark Tac Computer: A Sweet Way To Encourage Kids To Stay in Mathematics," by John W. Bailey and Kylene E. Norman; (2) "Using the Number Line To Factor Quadratics," by D. Irene Fell Rodgers; (3) "Numerical Integration: Surprises from the Trapezoidal Rule," by Travis Thompson; (4) "Bilateral Symmetry of the Hypergeometric Distributions," by Sandra M. Pulver; (5) "A Recursive Algorithm for Solving Linear Diophantine Equations," by Gene P. Blandford; (6) "An Induction Proof of the Inequality of the Weighted Geometric and Arithmetic Means," by Terrence Brenner; (7) "Pre- and Post-Instructional Strategies for the College Developmental Mathematics Class," by Joannis Kelly Flately, Dennis Gittinger, and Adele Ducharme; amd (8) "Reflections on the Studies of Comparisons of U.S. Students and Students in Other Countries," by James Baldwin. (AYC)

ED348071

"PERMISSION TO REPRODUCE THIS MATERIAL IN MICROFICHE ONLY HAS BEEN GRANTED BY

K.J. Smith

TO THE EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)."

U.S. DEPARTMENT OF EDUCATION
Office of Educational Research and Improvement
EDUCATIONAL RESOURCES INFORMATION CENTER (ERIC)

☐ This document has been reproduced as received from the person or organization originating it.

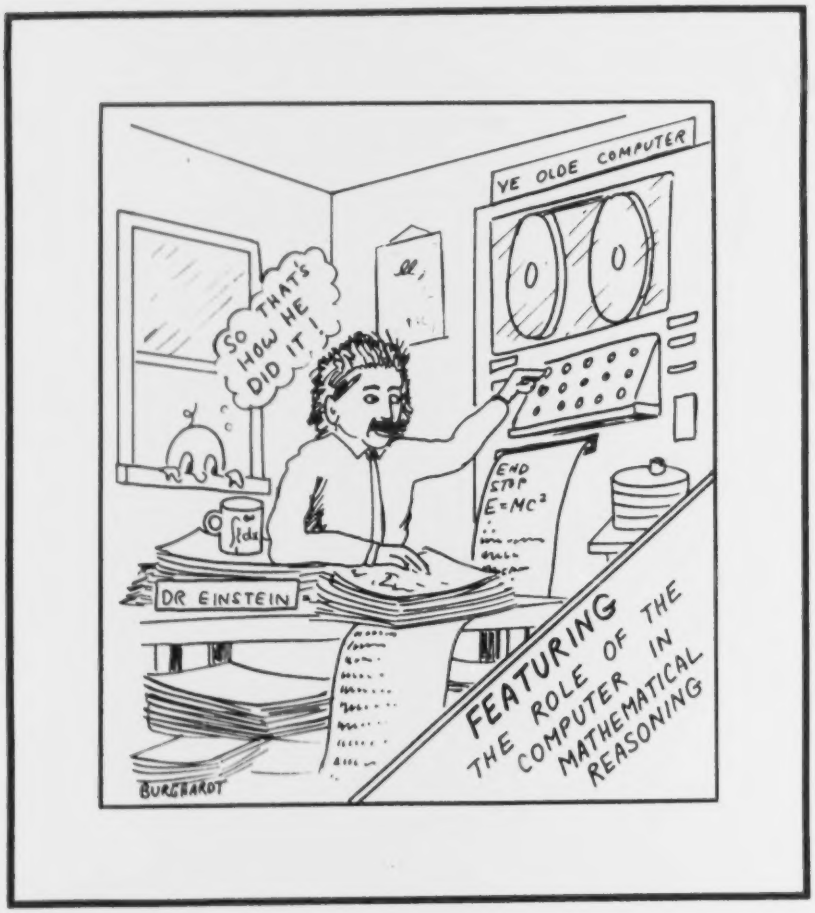
X Minor changes have been made to improve reproduction quality.

• Points of view or opinions stated in this document do not necessarily represent official OERI position or policy.

THE AMATYC REVIEW

Published by the AMERICAN MATHEMATICAL ASSOCIATION OF TWO-YEAR COLLEGES

VOLUME 9 NUMBER 1 FALL 1987



ALSO IN THIS ISSUE:

— QUESTIONBOOKS: Using Writing to Learn in Mathematics
— A Unified Approach to the Circular and Hyperbolic Functions
— Apparent Aging: An Explanation of a Common Paradox

IC 880 364

BEST COPY AVAILABLE

The Official Journal of the

American Mathematical Association of Two-Year Colleges

PURPOSE

AMATYC remains committed to its original purposes:

- To provide a national forum for the exchange of ideas
- To further develop and improve the mathematics education of students of two-year colleges
- To coordinate activities of affiliated organizations on a national level
- To promote the professional development and welfare of its members

The AMATYC Review provides an avenue of communications for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

SUBMISSION OF MANUSCRIPTS

MANUSCRIPTS MUST CONFORM TO THE FOLLOWING FORMAT:

They must be typewritten on one side and double-spaced on 8½" x 11" paper. Five copies of each manuscript, which has not been published before, should be submitted to the editor. If the manuscript contains any photographs, they should be black and white glossy prints. The first page of the manuscript should contain the article's title, all the authors' names and their institutional affiliation along with the name and address of the person to whom all correspondence should be mailed. Authors are advised to consult the *Publication Manual of the American Psychological Association* (3rd ed.). A guideline for authors is available from the editor.

Manuscripts should be sent to: Don Cohen, Editor, *The AMATYC Review*, SUNY College of Agriculture and Technology, Cobleskill, NY 12043. Any manuscript not accepted for publication will be returned.

ADVERTISING

For information concerning advertising rates and schedules, write to Eleanor Young, Sinclair Community College, Dayton, OH 45402 or telephone (513) 226-2789.

EDITORS

Editor: Don Cohen, SUNY College of Agriculture and Technology, Cobleskill, NY 12043, (518) 234-5116.

Production: Paul Dudenhefer, State Technical Institute at Memphis, TN 38134.

Advertising: Eleanor Young, Sinclair Community College, Dayton, OH 45402.

Problem Section: Michael W. Ecker, 129 Carol Drive, Clarks Summit, PA 18411.

Book Review: John J. Edgell, Jr., Southwest Texas State University, San Marcos, Texas 78666.

Computer Software Review: Shao Mah, Red Deer College, Red Deer, Alberta, Canada T4N5H5.

Artwork: Robert Burghardt, Rockland Community College, Suffern, NY 10901.

PUBLICATION

The AMATYC Review is published twice a year in September and February.

The AMATYC Review Fall, 1987 Volume 9, Number 1

Editorial ..... p. 5
Letter to the Editor ..... p. 6

VIEWPOINT
The Role of the Computer in Mathematical Reasoning ..... p. 8
by Bill Leonard and Harris Schultz

MATHEMATICAL EXPOSITION
The Trigonometric Functions and Arcs ..... p. 13
by Richard F. Maruszewski
Apparent Aging: An Explanation of a Common Paradox ..... p. 18
by Joseph Browne
Hindu Romance with Quadratic Functions ..... p. 23
by Gurcharan Singh Bhalla
A Unified Approach to the Circular and Hyperbolic Functions ..... p. 31
by Mary Coughlin

SHORT COMMUNICATIONS
"Don't Destroy the Butterfly" ..... p. 37
by William Walworth
An Unexpected Way to an Expected Value ..... p. 38
by Richard F. Maruszewski and Lee Parsons

MATHEMATICS EDUCATION
QUESTIONBOOKS: Using Writing to Learn in Mathematics ..... p. 41
by Agnes Azzolino and Robert G. Roth
What Differential Equations Books Lack ..... p. 50
by George E. Abdo

REGULAR FEATURES
Educational Reflections ..... p. 53
by Gregory D. Foley
Pandora's Rectangular Parallelipiped ..... p. 60
by Leslie R. Tanner
Book Reviews ..... p. 62
by John J. Edgell
Problem Section ..... p. 67
by Michael W. Ecker
Computer Software Section ..... p. 72
by Shao Mah

Editorial

The gestation period is about to end. My first issue as editor of *The AMATYC Review* is ready to go into production. I am proud of it and I hope that you will be proud of it. When I was appointed editor in July, 1986, our departing editor, Jay Huber, had already prepared the September, 1986, and February, 1987, issues. My job was to start working on the September, 1987, issue. During this initial effort I attempted to keep in mind the primary purpose of our journal:

> *To provide an avenue of communication for all mathematics educators concerned with the views, ideas, and experiences pertinent to two-year college teachers and students.*

Fortunately, most of the people who worked with Jay have remained in their positions; hence, the editorial transition has gone smoothly. Paul Dudenhefer and Eleanor Young remain as the production and advertising managers, respectively; and Michael Ecker, John Edgell and Shao Mah continue as department editors. On the other hand, several new people and features have been added. Bob Burghardt is our art editor and will design and draw a lively cover for each issue. On the inside, we have added two regular columns. Greg Foley will write on what he sees as trends in mathematics education in his "Educational Reflections" while Les Tanner will add some humor to our reading with his "Pandora's Rectangular Parallelipiped."

What may not be so apparent are the operational changes that have taken place. We now have an eight-member editorial panel which will help me to select and develop manuscripts for publication. Each article that is submitted is reviewed by two regular reviewers and one member of the panel. The panel member then prepares a summary recommendation on publication. It has been my experience that few articles are summarily rejected. Usually, articles are either accepted "as is" or with suggested minor revisions; or, they may be returned to their authors with extensive suggestions on how they might be improved before they are resubmitted. My primary function and that of the editorial panel is to nurture the development of articles that meet the needs of our membership.

In closing, let me encourage you to get involved with *The AMATYC Review*. This is your journal and only you can guarantee its success. While I am writing this introductory editorial, I would like to invite you, our readers, to write future editorials and to write letters in response to opinions that are presented. Also, share your ideas with your colleagues throughout AMATYC by submitting articles for publication and volunteer to be one of our regular manuscript reviewers. I will be happy to send our guidelines to any of you prospective authors. I am looking forward to hearing from you.

Don Cohen, Editor

Letter to the Editor

Editor's Note: Do you sometimes have the feeling that your classes get dull about half-way through the semester? Perhaps you can perk-up your students' interest by inviting a guest lecturer to your college. Les Tanner, president of the Idaho Council of Teachers of Mathematics and professor of mathematics at the College of Idaho, appears to have "perked-up" at least one of Gordon Hoagland's students while delivering a lecture entitled "Some Neat Stuff in Mathematics" at Ricks College during the Fall, 1986, semester.

Dear Editor,

A lecturer always wonders, but seldom knows, what impact his words have on his listeners. The following essay was written as part of a class assignment by one of the "captive audience" at a guest lecture and was forwarded to me by Professor Hoagland. I was not only flattered, but felt that others might be interested in reading it. Its words and message really "made by day," and give me reason to believe that not all I say to students simply goes in one ear and out the other.

Sincerely, L. R. Tanner

A NUMBER WONDERLAND NIGHTMARE

A familiar thought probably ran through all of our heads at the grand declaration of a way to achieve some "extra credit points" (you know, those three words considered equivalent to swear words by a professor) by attending a mathematics lecture. What were we to expect to get out of this? Were we even going to understand this topic? Oh, what the heck. I had **some** spare time. The worst it could do was keep me awake.

Actually, and probably surprisingly to most, the lecture was **good.** To think we learned something and at the same time enjoyed it! Learning has always brought such torture to the mind, how could we enjoy it? However weird it seemed, all of us were caught up in the realm of mathematics with smiles on our faces. Note: This is to remain a secret among those who attended, or else!

Professor Tanner lectured on three facets of mathematics: large numbers, number patterns, and the mind-and-paper twisting topic of flexagons.

How big is a "large" number? Can you describe the size of 1,000? Probably so. I have been in a stadium at NAU with nearly 10,000 people in the same vicinity. Now to think that is only a small fraction of the number of people in Arizona, the U.S., the world . . . of all Heavenly Father's children. Now you are getting the idea of a **large** number.

He gave the names of a few common large numbers; billion, trillion, quadrillion, quintillion, etc. Other examples, included vigintillion, 10^{63} and centillion, 10^{303}. Granted, these **are** large numbers; however, can you describe, let alone comprehend, even the number **million**? He used a chart of one million dots that stretched nearly across the room, blowing our minds. No one really realizes how big a large number actually is.

Other examples he used from his computations of objects in reality were: one billion $1 bills would cover 70 miles down a highway; one trillion $1 bills covers 70,000 miles; and one light year is five-plus trillion miles! Then there was his favorite number, one I had never heard of and wondered if he was playing caveman: a googol, or 10^{100}. Not a very small number unless you compare it to the googolplex or 10^{GOOGOL}. Okay, we have just now hit the twilight zone and my mind has gone to putty. On to other things.

Do you remember those patterns of numbers where you had to figure out the missing number? The easier and more boring part of math, right? Ah, you have not been through nature with Fibonacci. An instrument to measure the number of fibers in a specimen you may think, but then, that is what you get for thinking. The Fibonacci sequence is so impressive that it has its own society of researchers who look for this number sequence in ordinary, simple life. The sequence 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, ... is formed by adding the first two numbers to get a third, adding the third and second for a fourth, fourth and third for a fifth, etc. Big deal, right? Anyone can add. The amusing and amazing part is where this sequence has been found; the reproduction of bees, and the number of spirals of seeds on a sunflower and the petals on a pinecone are a few samples. And to think I have eaten sunflower seeds all my life and have never known this simple fact! Perhaps there is more in life than adding numbers together. I have been deprived! I think I will demand my money back from a few math teachers.

Before I was allowed to let my fury loose, my mind was taken back to the twilight zone with Mr. Gardner's flexagons. I thought large numbers were hard on the head. They are only overwhelming. However, flexagons are like trying to comprehend eternity, no beginning, no end. At least large numbers have a beginning and some sort of end, way, way out in number wonderland.

This is the exciting part; paper, glue, and scissors were in sight. Object lesson, my favorite! By cutting a strip of paper, gluing its ends together with a few twists here and there, you could walk along and end up at the same spot but on the opposite side of the paper. A piece of paper with one side and one edge? This is going to take some thought — or blind faith if your brain is fizzled by now. Putting some more do-dads to it, you can end up with a hexagon-shaped object. That is nice. Here is the catch. Turn it inside out a few times to find it has three faces! Well, let's call this a tri-hexa-flexagon (three faces, hexagonal shape, flexible). How handy! You have heard of the face that could launch a thousand ships? How about the flexagon that can turn twelve faces?

By this time my mind had been drained to nothing. All these "fun" little gadgets and wonderful new perspectives are probably going to haunt me tonight in a **Number Wonderland Nightmare!** All this to kill two birds with one stone — two extra credit bonuses to get an audience. What will they think of next? Well, good-night, sleep tight, and don't let the large numbers bite!

Kristine Crowther, Student
Ricks College,
Rexburg, ID 83440

VIEWPOINT

The Role of the Computer In Mathematical Reasoning

by Bill Leonard and Harris Shultz
California State University, Fullerton
Fullerton, California 92634

Bill Leonard is the author of twenty-three published articles and two books and has been an invited speaker at mathematics conferences and symposia over 170 times, including keynote speaker for more than one dozen different organizations.

Harris Shultz is the author of thirty published papers and is Co-Principal Investigator of the Southern California Mathematics Honors Institute and the Mariana Islands Mathematics Institute, both funded by the National Science Foundation.

In 1976 Professors Kenneth Appel and Wolfgang Haken published a proof of the "four-color" problem. For at least 125 years it had been conjectured that any map on a plane can be colored with no more than four colors so that no two adjacent countries have the same color. The proof by these two University of Illinois mathematicians was highly publicized for two major reasons. First, the problem was very well known, even among non-mathematicians. Second, the proof was dependent upon over 1000 hours of computer calculations and decisions. In other words, the proof presented by Appel and Haken cannot be carried out, or even verified, by someone working only with pencil, paper and human intellect.

This proof of the four-color problem has generated considerable controversy regarding the role of the modern computer in "pure" mathematics (see Mitchem, 1981). The work of Appel and Haken is an example of the verification of a conjecture, which, as of this writing, would still not be verified without modern technology. However, there are other degrees to which the computer can be used in the analysis or proof of mathematical theorems. For example, beginning with any natural number, we can generate a sequence recursively by defining the term immediately succeeding a number N to be $N/2$ if N is even and $3N+1$ if N is odd. As an illustration, the number 7 will generate the sequence 7, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1. It has been conjectured that every such sequence will contain the number 1 and the use of computers to verify thousands of cases has strengthened our confidence in the conjecture. However, to this day, no proof of any kind is known.

An example of a somewhat different role the computer can play in mathematics is provided by the question, "If two points are chosen at random from a unit square, what is the expected distance between them?" In our first attempt at an answer, we were somewhat intimidated by the rather complicated calculus and analysis involved. So we turned to the computer to randomly generate hundreds of cases and compute the desired average. That is, we asked the computer to randomly select two

points in the unit square, compute the distance between them, repeat the process many times, and, finally, compute the average of these distances. A run of 10,000 iterations yielded an average of 0.516096605. With this approximation in hand, we were more motivated to determine the "real" answer. Our determination required the evaluation of the quadruple integral

$$\int_0^1 \int_0^1 \int_0^1 \int_0^1 \sqrt{(u-s)^2 + (v-t)^2}\ ds\ dt\ du\ dv.$$

A somewhat complicated combination of various unpleasant integration techniques yields the result

$$\frac{2 + \sqrt{2} + 5 \ln (1 + \sqrt{2})}{15}.$$

It is gratifying to note that this is approximately 0.521405433.

We have outlined above three ways in which the computer can become very much a part of mathematical reasoning. It can provide an otherwise unobtainable proof of a theorem, it can strengthen one's confidence regarding the truth of a conjecture, or it can indicate a numerical result later to be proved analytically.

We now demonstrate another way in which the computer can be part of the process of mathematical reasoning. Beginning with any natural number, sum the squares of its numeral's digits. The process is continued with this second number and so forth until a sum of either 4 or 1 is reached.

Consider the number 1940. We generate the sequence for this number as follows:

$$1^2 + 9^2 + 4^2 + 0^2 = 98$$
$$9^2 + 8^2 = 145$$
$$1^2 + 4^2 + 5^2 = 42$$
$$4^2 + 2^2 = 20$$
$$2^2 + 0^2 = 4.$$

The number 1959 generates the sequence 188, 129, 86, 100, 1. Ross Honsberger (1970), in his charming book, **Ingenuity in Mathematics,** states without proof that the above process will always yield the number 1 or the number 4. We prove the above result as follows.

The BASIC program below tests the natural numbers from 1 to 324 and shows that each sequence generated indeed does contain either 1 or 4.

```
10  FOR I = 1 TO 324
20  N = I
30  PRINT N; " ";
40  IF N = 1 OR N = 4 THEN 200
50  A = INT (N/10)
60  B = INT (N/100)
70  C = A - 10*B
80  D = N - 10*A
90  N = B*B + C*C + D*D
100 GOTO 30
200 PRINT
210 NEXT I
220 END
```

The largest four digit number, 9999, generates 324 when the squares of its digits are summed. Therefore, numbers less than 9999 will generate sums less than 324 on the first iteration, and we conclude that the process will yield a 1 or a 4 for all natural numbers having four or fewer digits.

We now need consider only those numbers having more than 4 digits. Let n be a number with k digits, where $k > 4$. It is easily seen that

$$10^{k-1} > 100k,$$

which implies that

$$10^{k-1} > 9^2k.$$

Since the sum of the squares of the digits of n is less than or equal to 9^2k in any k-digit number, and since $n \geq 10^{k-1}$, we see that the sum of the squares of the digits of n is less than n. Therefore, the sequence generated by any number having more than four digits will be strictly decreasing at least until a number having four or fewer digits is reached. However, we have already seen that a number having four or fewer digits will yield a 1 or a 4. This completes the proof.

Part, but not all, of the above proof was carried out by a computer. However, unlike the proof of the four-color theorem, the computer's work could be somewhat tediously done by human hand and intellect.

We do not claim authority to judge whether or not the proof by Appel and Haken of the four-color problem is indeed mathematically acceptable. However, we do contend that the growing importance of the computer in mathematical reasoning cannot be

ignored. As we have seen, it can motivate investigation, it can provide credence to a conjecture, and it can facilitate an otherwise tedious proof.

The proof given above is ideally suited for computer work since students may actually discover patterns by generating data and then analyzing the results from a theoretical point of view. We conclude with suggestions for two further areas of investigation which might appeal to students.

(1) Consider summing the cubes, rather than squares, of the digits. One will find, for example, that multiples of three always yield 153.

(2) A computer analysis relative to the squaring process of all numbers up to 2,140,000 indicates that the fraction of numbers leading to 1 rather than 4 seems to be converging on a number near .1416.

REFERENCES

Honsberger, R. (1970). *Ingenuity in mathematics*. New York: Random House.

Mitchem, J. (1981). On the history and solution of the four-color map problem. *The Two-Year College Mathematics Journal, 12*, 108-116.

Innovations In Learning... from H&H Publishing.

College Mathematics Review *3rd Edition*

by Robert Blitzer and Jack C. Gill, Miami-Dade Community College

- New revised and expanded edition includes chapter summaries and sample examinations
- Topics include Sets & Logic, Probability & Statistics, Algebra & Number Theory, Informal Geometry and Basic Arithmetic
- Brief, expository format, hundreds of illustrative problems, and over 1000 class-tested homework problems
- Third Ed., 1987, 349 pages, ISBN 0-943202-25-6, $21.95

Competency In College Mathematics *4th Edition*

by Jack C. Gill and Robert Blitzer, Miami-Dade Community College

- Covers all traditional liberal arts math topics and includes the most extensive logic section of any book of this type
- Review exercises in each exercise section, many tests, all answers provided
- Fourth Ed., 1986, 609 pages, ISBN 0-943202-19-1, $23.95

H&H Publishing Company

2165 Sunnydale Blvd., Suite N, Clearwater, FL 34625-1211, (813) 442-7760

MATHEMATICAL EXPOSITION
THE TRIGONOMETRIC FUNCTIONS AND ARCS

by Richard F. Maruszewski, Jr.
United States Naval Academy
496 Century Vista Drive
Arnold, MD 21012

Richard F. Maruszewski, Jr., is currently on recall to active duty with the Navy, teaching mathematics at the United States Naval Academy in Annapolis, Maryland. While at the Academy, he is on leave of absence from Jamestown Community College in Jamestown, New York. He received his BS and MS in Mathematics from Marquette University and his Ph.D. from the University of Wisconsin in Milwaukee.

USNA
U.S. Naval Academy

JCC
Jamestown Community College

1. Introduction

When working with the trigonometric functions, did you ever ask yourself: Why do we refer to the tangent of an angle when the inverse function, Arctangent, refers to an arc? Why doesn't our definition of "tangent" involve a tangent line? What does the word "sine" mean?

Webster's Dictionary (1960) offers the following definitions:

Sine: formerly, the perpendicular drawn from the extremity of an arc of a circle to the diameter drawn through the other extremity.

Secant: a straight line cutting a curve.

Note that the former definition of a sine refers to an arc, not an angle. Also the definition of a secant raises the question as to why the trigonometric definition of secant isn't somehow associated with such a cutting line. The following presentation, suggested by Professor James C. Abbott of the United States Naval Academy, gives answers to the above questions that include trigonometric functions defined in terms of the lengths of arcs on a unit circle. Clearly from the definitions above, the trigonometric functions were originally defined in a similar way. We hope that this presentation will offer an interesting alternative to the traditional one in the trigonometric section of a Calculus course.

2. The Wrapping Function and Standard Position

Swokowski (1976, pp. 181-86) defines the standard position of an arc as follows: Begin with a unit circle with a horizontal diameter and a tangent line to the right extremity of the diameter (See Figure 1.) The tangent line may now be scaled by using the radius of the unit circle. The origin is at the intersection of the diameter and the tangent, and the upward direction is positive. Our arc is now laid off in the counterclockwise direction on the circle starting at the origin. Therefore, given a length of arc, we first lay the arc down on the positive tangent axis and then "wrap"

the arc around the unit circle. We are now ready to define the trigonometric functions. We will begin with arcs which terminate in the first quadrant, but later all arcs will be considered. Using Figure 2, we define for x = length of arc BC.

(a) $\tan(x) = |BD|$

(b) $\sec(x) = |AD|$

(c) $\sin(x) = |CE|$

Note the following:

(a) The tangent of x is the length of a segment of a tangent line. (This definition is, of course, not new. For example, see Stein (1967, p. 83).)

(b) The secant of x is the length of a segment of a cutting line.

(c) The sine of x is the length of a sine as defined in Webster's.

When we turn to the cosine, cotangent, and cosecant, we are, of course, talking about the sine, tangent, and secant of the complement of our arc. Consulting Webster's again, we learn that the complement of an arc (That's right, an arc!) is the amount of arc that the given arc falls short of the arc with a central angle of 90 degrees.

We, therefore, produce Figure 3 with another tangent line for the complement of BC (arc CG) and define:

$\cos(x)$ = sine of the complement = $|CH| = |AE|$
$\cot(x)$ = tangent of the complement = $|GF|$
$\csc(x)$ = secant of the complement = $|AF|$

3. **Trigonometric Functions.**

Referring to Figure 3 and our definitions, we can now prove some of the basic trigonometric identities.

(3.1) $\tan(x) = \sin(x)/\cos(x)$

PROOF: Triangle ACE is similar to triangle ADB. Thus
$|DB| / |CE| = |AB| / |AE|$
But $|AB| = 1$ because the radius of the circle is of length 1. So
$|DB| = |CE| / |AE|$ or
$\tan(x) = \sin(x)/\cos(x)$

(3.2) $\cos(x) = 1/\sec(x)$

PROOF: Using the same triangles as in (3.1), $|AE| / |AB| = |AC| / |AD|$ or
$\cos(x)/1 = 1/\sec(x)$

(3.3) $\sin^2(x)+\cos^2(x) = 1$

PROOF: By the Pythagorean Theorem,
$|AC|^2 = |CE|^2 + |AE|^2$

(3.4) We can now, of course, derive other identities such as dividing both sides of (3.3) by $\cos^2(x)$ to obtain $\tan^2(x)+1 = \sec^2(x)$.

4. The Derivatives of the Trigonometric Functions

Having touched lightly on trigonometric identities, we turn to the calculus of the trigonometric functions. The following are examples of the development of the derivative formulas.

(4.1) If $y = \tan(x)$ then $y' = \sec^2(x)$

PROOF: We refer to Figure 4.

(a) $f'(x) = \lim_{\Delta x \to 0} \Delta y/\Delta x$

(b) x = the length of arc BC
Δx = the length of arc CG
$y = |DB|$
$\Delta y = |ID|$

(c) $\Delta y/\Delta x = (\Delta y / |HD|)(|HD| / |FC|)(|FC| / \Delta x)$
But triangle AFC is similar to triangle AHD, so
$|HD| / |FC| = |AD| / |AC| = \sec(x)/1 = \sec(x)$
And triangle ABI is similar to triangle DHI, so
$\Delta y / |HD| = |ID| / |HD| = |AI| / |AB| = \sec(x+\Delta x)/1$
Finally, note that $\lim_{\Delta x \to 0} |FC| / \Delta x = 1$

So $f'(x) = \lim_{\Delta x \to 0} (\sec(x+\Delta x))(\sec(x))(|FC| / \Delta x) = \sec^2(x)$

(4.2) If $y = \text{Arctan}(x)$, $y' = 1/(x^2+1)$

PROOF: $y = \text{Arctan}(x)$
$x = \tan(y)$
Differentiating implicitly,
$1 = (\sec^2(y))(y')$
$y' = 1/\sec^2(y) = 1/(\tan^2(y)+1)$
$= 1/(x^2+1)$

The derivatives of the other trigonometric functions and their inverses may be developed by using the trigonometric identities and the above.

5. The Trigonometric Functions of Any Arc

The preceeding dealt with arcs in standard position whose central angles were in the first quadrant. The definitions extend naturally to any arc in standard position as follows. Start with the horizontal diameter again and a tangent axis scaled as previously described. Replace the sine with a sine axis with the origin at the intersection of the axis and the diameter and again with up as positive. Replace the secant with a secant axis with origin at the center of the circle and positive in the direction of the intersection of the axis with the terminal extremity of the arc. Figures

5, 6, and 7 are illustrations for the arc BC in different quadrants. The scale for the axes comes from the radius of the circle which is of length one. The secant of x is the value of the point on the secant axis where it intersects with the tangent axis. This same point on the tangent axis is the value of the tangent of x. The sine of x is likewise the value of the point on the sine axis where it intersects with the terminal extremity of BC. Note in Figure 5 the sine is positive, and the tangent and secant are negative. In Figure 6 the tangent is positive while the secant and sine are negative. Finally in Figure 7, the secant alone is positive. In all cases, triangles like the ones in the earlier proofs can be used again to attain the desired results.

6. Summary

Obviously, much more can be done in all of the sections of this article, but it was a good beginning in what we set out to do. We gave a presentation of the trigonometric functions defined in terms of the length of arcs, and we defined the functions in settings corresponding to the meanings of the words used to name the functions.

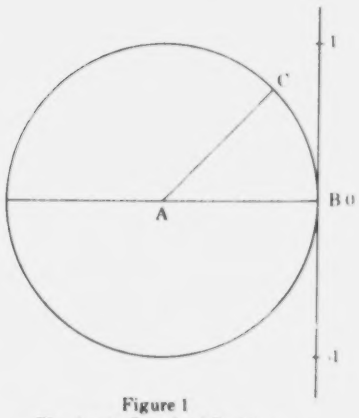


Figure 1
The Arc in Standard Position

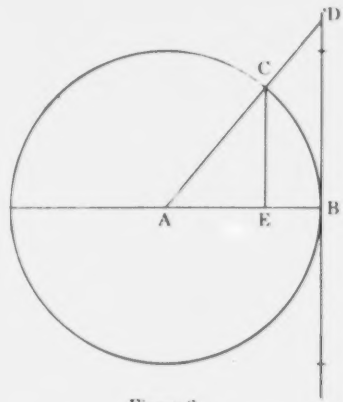


Figure 2
The Trigonometric Functions

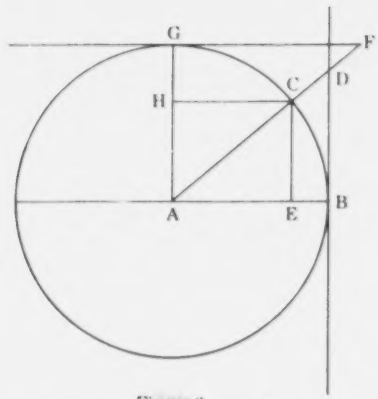


Figure 3
The Complementary Arc

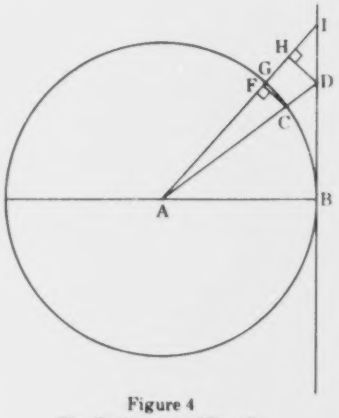


Figure 4
The Derivative of Tan(X)

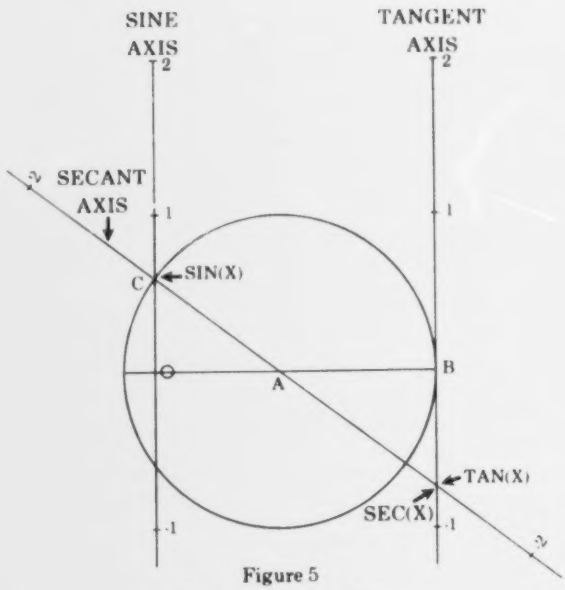


Figure 5
Second Quadrant Arc

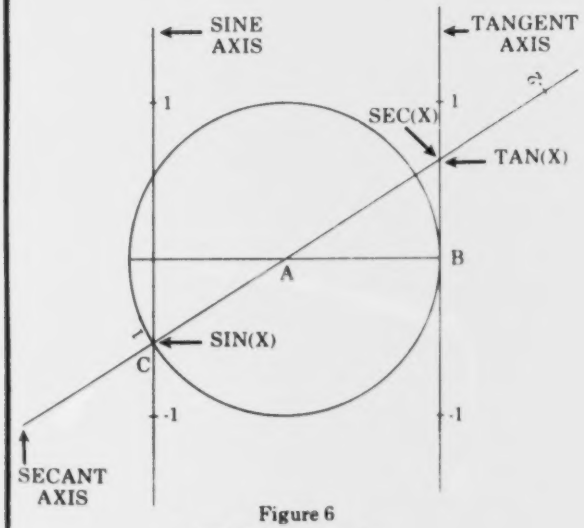


Figure 6
Third Quadrant Arc

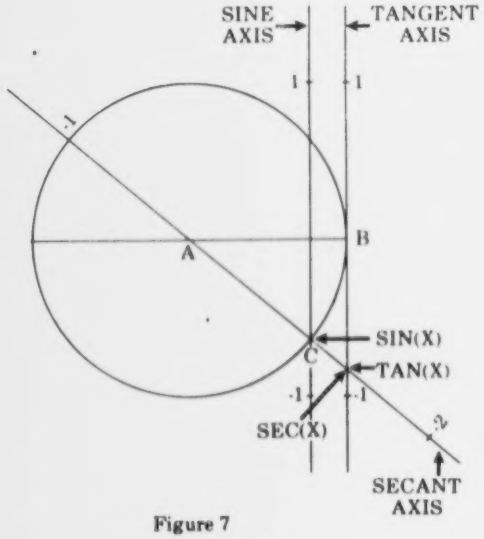


Figure 7
Fourth Quadrant Arc

References

1. Stein, S. (1967). *Calculus and analytic geometry.* New York: McGraw Hill.
2. Swokowski, E. (1976). *Algebra and trigonometry.* Boston: Prindle, Weber & Schmidt.
3. *Webster's new collegiate dictionary.* (1960). Springfield, MA: G. & C. Merriam Co.

BEST COPY AVAILABLE

Apparent Aging:

An Explanation of a Common Paradox

by Joseph Browne
Onondaga Community College
Syracuse, NY 13215

Joe is very active in the mathematical community. In addition, to being the president of the New York State Mathematics Association of Two Year Colleges, he serves as the problem editor of the *Mathematics and Computer Education Journal.*

Editor's Note: Professor Browne occasionally suffers from a condition known as "tongus in cheekus." In order to protect themselves from catching this disorder, readers are urged to place their tongues firmly in their cheeks before reading this article.

The observation has been made many times, both inside and outside the scientific community, that years seem to get shorter as we get older. Birthdays seem to get closer together, vacations don't last as long, and children are married and gone shortly after we realize they've hit puberty. References to this illusion can even be found in song lyrics, such as "it's later than you think," and "the days grow short as you reach November."

One theory, advanced early in the formal study of this phenomenon, was that the length of the year actually was getting shorter. A massive body of data contradicting this theory has caused its proponent, T. L. Freeman, to admit that the theory must be discarded (see Freemen, 1983). Freeman went on to propose a second theory which gives a more satisfactory explanation, but is marred by sloppy mathematical techniques and seemingly arbitrary assumptions. It is the purpose of this paper to clean up Freeman's reasoning and put his admirable theory on a firm foundation.

The basis of Freeman's theory is that the frame of reference for judging time intervals is constantly changing. A year, for example, represents 20% of the life of a 5 year old child, but only 2% of the life of a 50 year old adult. Thus, the child perceives the year as being much longer, indeed, ten times as long. The general principle can be stated quantitatively as the **Freeman Hypotesis:** The actual rate at which time appears to pass is inversely proportional to the observer's age.

Let a = the observer's actual age, and

t = the observer's apparent (to him/herself) age.

Then the law can be restated, mathematically, with the following differential equation:

$$\frac{dt}{da} =$$

where k is the constant of proportionality. Therefore,

$$dt = k \frac{da}{a}$$

and

$$t = k \ln a + c.$$

Next, we determine the constant of integration, c. The obvious initial condition would be that actual age, a, and apparent age, t, are both zero at birth. A problem arises, though, because the logarithm of zero is undefined. Freeman glosses over this matter in a vague footnote. The problem is not insurmountable, however. We can temporarily use the Asian custom of considering a child's age to be unity at birth. After determining the constant, we then make a simple translation to put the equation back into the age system to which we are accustomed. Letting both a and t be unity, we get

$$1 = k \ln 1 + c,$$

and since $\ln 1 = 0$, we have $c = 1$. Our equation now reads

$$t = k \ln a + 1.$$

But since our ages are one less than the Asian ages, we replace a and t by a+1 and t+1, respectively, yielding

$$(t + 1) = k \ln (a + 1) + 1$$

which simplifies to

$$t = k \ln (a + 1).$$

Now, we must determine the constant of proportionality, k.

This is another place where Freeman's methods leave something to be desired. He assumes that the rate of passage of time seems about right at (actual) age 20. This was his own arbitrary judgment, and we would obviously get widely varying values if we asked many different people. It would clearly be more desirable to use some condition which has broader acceptance and greater consistency. We submit that the following statement about human nature provides just the sort of universally acknowledged standard which is needed: All 14 year olds think they know everything. (If you are not familiar with 14 year olds, just ask a few parents.) Since a person's intellectual powers are generally considered to peak at about age 55 (no,

that is not the author's age), we have here an indicator that a = 14 corresponds to t = 55. Putting these into the last equation, we get

$$55 = k \ln 15$$

$$k = \frac{55}{\ln 15} = 20.3.$$

Our law, then, can be expressed as

$$t = 20.3 \ln (a + 1).$$

Some typical values are given in the table below and displayed in the graph in Figure 1.

Actual age	Apparent age	Actual age	Apparent age
0	0	50	79.8
5	36.4	55	81.7
10	48.7	60	83.5
15	56.3	65	85.1
20	61.8	70	86.5
25	66.1	75	87.9
30	69.7	80	89.2
35	72.7	85	90.4
40	75.4	90	91.6
45	77.7	95	92.7
		100	93.7

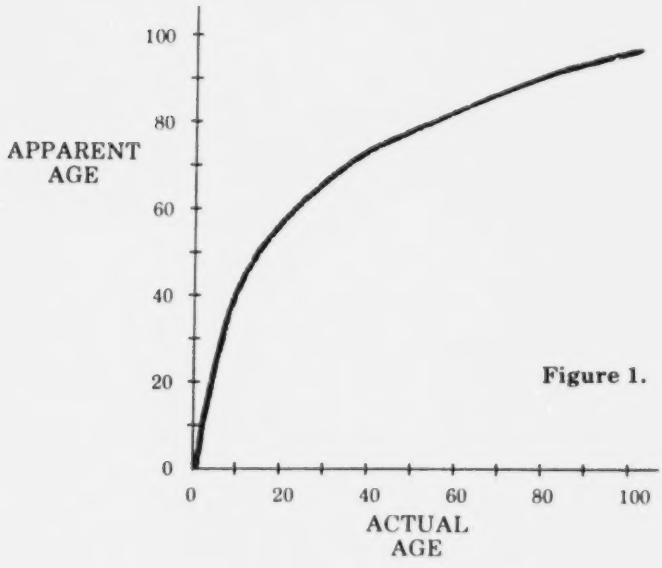


Figure 1.

As we can see from the table and the graph, apparent age rises quickly while we are young, and later almost levels off. Explanations for several seeming paradoxes can be found in these figures. As a child, the school year seems to last forever. The formula shows that to a five year old, the nine months of kindergarten appear to last 2.4 years. Do your weekends seem to be getting shorter? If you are 45 years old, the apparent length of a two day weekend is down to 21 hours, 11 minutes (and shrinking). Have you noticed that you communicate better with your parents now than you did as a teenager? The so called generation gap narrows naturally as the generations involved get older.

Notice that the actual age and apparent age are equal at about 92. One would suspect that there is some significance to this crossing. Life expentancy has been increasing as medical knowledge has increased, but it is unreasonable to expect it to rise without bound. Might it be that this "magic" moment when actual age and apparent age are equal designates nature's ultimate life expectancy?

It would be naive to assume, as Freeman evidently does, that the relationship between actual and apparent age is always continuous. The analysis above must be regarded as merely the ideal pattern that life would follow if nothing unusual happened. It is common knowledge that a person's age takes a jump when his/her first child is born, when attending a class reunion, or when a death occurs to someone close. On the other hand, one feels younger when "elderly" athletes like Phil Niekro or Kareem Abdul-Jabbar have a good day, when an attractive member of the opposite sex gives us the eye, or when a sabbatical leave is granted. When events like these occur, the general pattern of aging does not change, but there are jump discontinuities in the curve. Mathematically, these adjustments can be accomplished by the addition of appropriate Heaviside functions. The plot of an actual life might look more like the graph in Figure 2.

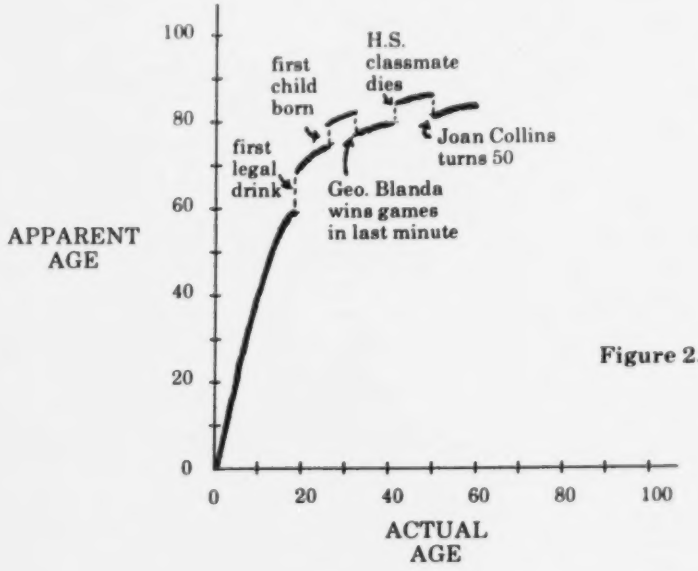


Figure 2.

In conclusion, we have shown that the rates of actual and apparent aging are related in a simple way, and that this relationship provides explanations for many perceived paradoxes of aging. Further research should be directed at how to exploit the nonlinearity of apparent aging in ways that would benefit human-kind both socially and economically.

REFERENCE

Freeman, T. L. (1983). Why it's later than you think. In George H. Scherr (Ed.), *The best of the journal of irreproducible results*. New York: Workman Publishing.

HINDU ROMANCE WITH QUADRATIC EQUATIONS

by Dr. Gurcharan Singh Bhalla
Department of Mathematics
Bronx Community College
Bronx, NY 10453

Bronx Community College
of the City University of New York

The **Sulvasutras** (800 B.C.), the Brahmanical manuals on Vedic geometry, suggest that Hindus used quadratic equations in one variable in the construction of their religious altars (vedis). Like other branches of knowledge and social, economical, and political institutions, Hindu mathematics was under the mighty sway of religion, its codes and rituals. It was the altar-building ritual of the Hindus that motivated the birth, growth and the development of the subject.

Two well-preserved problems from **Sulvasutras** show that Hindus employed quadratic equations to enlarge or reduce their religious altars. The first problem is that of Great Altar (the Maha-Vedi). The altar is trapezoidal in shape with bases and altitude of 24, 30, and 36 units respectively and an area of 972 sq. units (Figure 1).

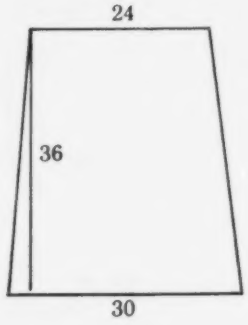


Figure 1

According to one prescription, the area of the altar was to be increased to (972 + a) sq. units. To achieve the enlargement of the area, the dimensions of the altar were increased proportionately to 24p, 30p, and 36p, with an area of 972 p^2 sq. units. Thus:

$$972\ p^2 = 972 + a$$

which is a pure quadratic equation in the sense that there are no linear terms in p. The solution of this equation provides information on the modifications required to achieve the increase in the area of the altar.

The second problem from the **Sulvasutras** deals with the falcon-shaped altar (the Syena-Cit). The altar consists of: a square body of 4 square units, two rectangular wings of 1 unit by 1 1/5 units, and a rectangular tail 1 unit by 1 1/10 units, with an area of 7 1/2 square units (Figure 2).

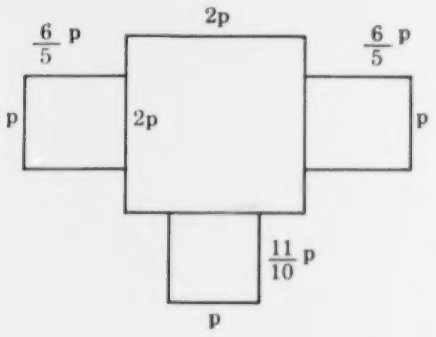


Figure 2

Two religious plans were in place for this altar:

1. The area of the altar was to be increased by a square units when the dimensions of the altar were increased proportionately.

The area of the modified altar is given by:

$$\text{Area} = 4p^2 - 2\left(\frac{6}{5}\right)p^2 + \frac{11}{10}p^2$$
$$= \frac{15}{2}p^2$$

This area must be equal to $\frac{15}{2} + a$. Hence:

$$\frac{15p^2}{2} = \frac{15}{2} + a$$

which is again a pure quadratic equation.

The second religious plan for the falcon-shaped altar resulted in an impure quadratic equation. According to this plan, the area of the altar was again increased by a square units. To do this, however, only the integral parts of the dimensions of the altar were to be modified and the fractional parts were to be kept untouched as indicated in Figure 3.

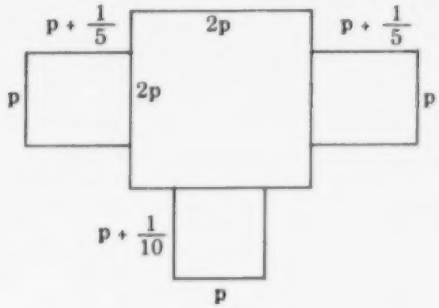


Figure 3

The area of the modified altar was given by:

$$\text{Area} = 4p^2 + 2p\,(p + \tfrac{1}{5}) + p\,(p + \tfrac{1}{10})$$

$$= 7p^2 + \tfrac{1}{2}\,p$$

This area must equal $\frac{15}{2} + a$. Hence:

$$7p^2 + \tfrac{1}{2}\,p = \frac{15}{2} + a$$

which is an impure quadratic equation.

These two problems from **Sulvasutra** suggest that Hindu scholars of the Vedic times were familiar with quadratic equations. The illustrations described above also bear out the important facts that the problems arose out of necessity and were not merely mathematical curiosities. The sacrificer of the falcon-shaped altar believed that after death he would become a falcon and soar high to the heavens. Mathematics was motivated by religious needs, and the Hindu mathematicians had a strong interest in the advancement of the subject.

With the dawn of the Christian era, new demands were placed on the subject, and the Hindu scholars were called upon to invent accurate procedures to chart out the heavens. This was the golden age of Indian astronomy. During this age, India was blessed with a succession of first-rate astronomer-mathematicians who met the challenge of the time and brought India to new frontiers of knowledge.

Aryabhata I, born in 476 A.D., was the first of these great Hindu scholars. In 499 A.D., he wrote a book entitled **Aryabhatiya,** which included a chapter on mathematics. The author used quadratic equations (a) to find the number of terms of an arithmetic progression, and (b) to solve a problem involving interest.

Aryabhata's rule for finding the number of terms of an arithmetic progression stated:

> "Multiply the sum of the progression by eight times the common difference; add the square of the difference between twice the first term and the common difference; take the square root of this; subtract twice the first term; divide by the common difference; add 1; divide by two. The result will be the number of terms " (Clark, 1930, p. 35).

Symbolically, if s is the sum of arithmetic progression, d the common difference, and a the first term, then n the number of terms is given by:

$$n = \tfrac{1}{2}\left(\frac{\sqrt{8sd + (2a-d)^2} - 2a}{d} + 1\right)$$

Aryabhata's solution of interest problems indicates the use of quadratic equations:

> A certain sum of money p is loaned for a period of time t; the interest x on the amount p, is lent on interest again at the same rate for time T. I the sum of x and the interest on x is given. Then x is given by:

$$x = \sqrt{\left(\frac{pt}{2T}\right)^2 + \frac{PIt}{T}} - \frac{Pt}{2T}$$

To illustrate the working of the rule, the following example is taken from the works of Brahamagupta, the second great Hindu mathematician after Aryabhata I.

> The interest accruing on 500 for 4 months is lent out for 10 months at the same rate of interest. Find the rate of interest if the sum of interest on 500 and the interest on the interest is 78.

Here p=500, t=4, T=10, and I=78; therefore, x, the interest on 500 is 60 and the rate of interest is 3% per month.

The Hindus' romance with quadratic equations continued. The Bakshali Manuscript, excavated in 1881 at the village of Bakshali near the city of Peshawar (now in Pakistan), was a compendium of rules and illustrative examples in arithmetic, algebra, and geometry. The work was a mathematical commentary on mathematical treatises of ancient times that probably dated from the second to fourth century A.D. The following example, taken from the manuscript, dealt with arithmetic progression as well as quadratic equations:

> A certain person travels five yojanas (1 yojana = 8000 times the height of a man) on the first day, and three yojanas more on each successive day; another person who travels seven yojanas a day has a start of five days. In what time will they meet ?

The distance traveled by the first man is the sum of an arithmetic progression with first term 5 and common difference 3. If the two persons meet in $\underline{x}$ days, then the distance d_1 traveled by the first person is given by:

$$d_1 = \frac{x}{2}[(2)(5) + (x - 1)3]$$

$$= 1/2\,(3x^2 + 7x)$$

The second person travels 7 yojanas per day and has a start of 5 days implies that he travels for $(x+5)$ days. Therefore, the distance d_2 traveled by the second person is given by:

$$d_2 = 7(x + 5) = 7x + 35$$

Since the two persons meet, $d_1 = d_2$, which implies

$$3x^2 - 7x - 70 = 0$$

$$\text{or } x \doteq 6.14 \text{ days}$$

After the times of Aryabhata I, we sight the treatment of quadratic equations in the works of Brahamagupta. Born in 598 A.D., he is one of the most celebrated mathematicians of Hindu India. Brahamagupta added a new dimension to quadratic equations. Recognizing the existence of negative numbers, he unified the three forms of quadratic equations given by Diophantus ($ax^2 + bx + c = 0$, $bx + c = ax^2$, $ax^2 + c = bx$) to one form: $px^2 + qx + r = 0$. He gave two rules for the solution of the quadratic equation of the form: $px^2 + qx = r$:

(i) "The quadratic: the absolute quantities multiplied by four times the coefficient of the square of the unknown are increased by the square of the coefficient of the middle (i.e. the unknown): the square root of the result being diminished by the coefficient of the middle term and divided by twice the coefficient of the square of the unknown, is (the value of) the middle" (Datta and Singh, 1962, pp. 62-63).

Symbolically, for the quadratic equation $px^2 + qx = r$

$$x = \frac{\sqrt{4pr + q^2} - q}{2p}$$

(ii) "The absolute term multiplied by the coefficient of the square of the unknown increased by the square of half the coefficient of the unknown: the square root of the result diminished by half the coefficient of the unknown and divided by the coefficient of the square of the unknown is the unknown" (Datta and Singh, 1962, p. 62).

Symbolically for the quadratic equation $px^2 + qx = r$

$$x = \frac{\sqrt{pr + (\frac{q}{2})^2} - \frac{q}{2}}{p}$$

Another method for solving quadratic equations came from Sridharacarya, a distinguished mathematician of 10th century A.D. He solved quadratic equations by multiplying all parts of the quadratic equations by 4 times the coefficient of the square of the unknown term. To solve $ax^2 + bx = c$, Sridharacarya suggested four steps:

1. Multiply all parts of the quadratic equation by 4a to get

$$4a^2x^2 + 4abx = 4ac$$

2. Add b^2 to both sides

$$4a^2x^2 + 4abx + b^2 = 4ac + b^2$$

3. Take square root of both sides

$$(2ax + b) = \sqrt{4ac + b^2}$$

4. Transpose and divide by 2a

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}$$

While the procedure suggested by Sridhara was brilliant and novel, he, like his predecessors, did not use a negative sign before the radical. This was left to the mathematical ingenuity and prowess of Bhaskara II (c. 1114), who went beyond the Greeks, Brahamagupta and other Hindu mathematicians when he stated that the square of a positive or negative number is positive and that the square root of a positive number is two-fold — a positive number and a negative number. He further stated that a negative number is not a square number. He, along with other Hindu mathematicians, recognized the existence of irrational numbers and subjected them to the same procedures as other numbers. These facts advanced the theory of quadratic equations appreciably besides giving further impetus to the advancement and enhancement of mathematics in India.

The Hindu mathematicians' romance with quadratic equations, as documented in this article, continued for about 2000 years from 800 B.C. to 12th century A.D. They employed quadratic equations in numerous situations: in reducing and enlarging their religious altars, in charting the motion of heavenly bodies, in problems related to interest and commerce, and in some cases to solve problems posed merely for amusement. Their mathematical discoveries were not limited to working with quadratic equations but extended far beyond. They worked with indeterminate

linear and quadratic equations, and their work in these areas remained unparalleled and unsurpassed for many centuries.

References

1. Clark, W.E. (1930), *The Aryabhatiya of Aryabhata*, Chicago, IL: University of Chicago Press.
2. Datta, B., & Singh, A.N. (1962). *History of Hindu Mathematics*, Pt. 2, Bomabay: Asia Printing House.

CALL FOR REFEREES

Readers are invited to help the editor to evaluate manuscripts by becoming members of *The AMATYC Review* reviewer panel. Write to Don Cohen, SUNY Ag-Tech College, Cobleskill, NY 12043, for more information.

A Unified Approach to the Circular and Hyperbolic Functions

by Mary Coughlin
The University of Toledo
Toledo, Ohio 43606

Dr. Mary Coughlin has a Ph.D. in Mathematics Education from the University of Michigan. She is a member of the Mathematics department at the University of Toledo in Ohio.

Introduction

There are those who believe that the presentation of elementary mathematics from an advanced point of view may obscure rather than clarify the ideas presented. We believe, on the other hand, that such a presentation is useful since, almost invariably, it introduces new insights and allows students to see apparently unrelated topics as part of an integrated whole. An example of such a presentation is Felix Klein's discussion of the analogy between the circular and hyperbolic functions contained in his famous book, **Elementary Mathematics from an Advanced Standpoint.** In this paper we propose to give a version of this analogy which differs from Klein's in some respects and extends it beyond the point where he left it.

The Circular Functions

In his development, Klein (1932) begins with the unit circle $x^2 + y^2 = 1$. We shall follow his example but modify his presentation slightly. We consider the sector S bounded by the curve $x^2 + y^2 = 1$ and the radii from the origin to the points P(1,0) and R(x,y), $x > 0$, $y > 0$, on the curve. If θ is defined to be twice the area of this sector S, it is easily seen that θ is also equal to the length of arc along the curve between P(1,0) and R(x,y). This follows from the proportion

$$\frac{\text{area of sector}}{\text{total area}} = \frac{\text{length of arc}}{\text{circumference}}.$$

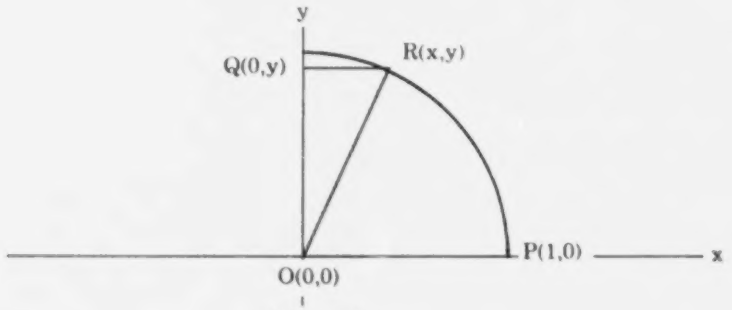


Figure 1.

It is usual nowadays to follow Klein and define $\sin\theta$ and $\cos\theta$ to be the values of the coordinates x and y such that $x = \cos\theta$ and $y = \sin\theta$. In his presentation Klein derives an integral representation for $\theta = \sin^{-1} y$ in the following way. The area of the curved region OPRQ shaded in Figure 1 is $\int_0^y \sqrt{1+t^2}\,dt$. Since the area of the triangle OQR is $\frac{1}{2}\,\overline{OQ}\cdot\overline{QR} = \frac{1}{2}\,yx = \frac{1}{2}\,y\sqrt{1-y^2}$, it follows that

$$\int_0^y \sqrt{1-t^2}\,dt = \frac{1}{2}\,y\sqrt{1-y^2} + \frac{1}{2}\,\theta.$$

Differentiating with respect to y, we obtain

$$\sqrt{1-y^2} = \frac{1}{2}\,y\,\frac{(-2y)}{2\sqrt{1-y^2}} + \frac{1}{2}\sqrt{1-y^2} + \frac{1}{2}\,\frac{d\theta}{dy}$$

and therefore $\frac{d\theta}{dy} = \frac{1}{\sqrt{1-y^2}}$ From this we conclude $\frac{dy}{d\theta} = \sqrt{1-y^2} = x$

and $\frac{dx}{d\theta} = -y$, that is, $\frac{d(\sin\theta)}{d\theta} = \cos\theta$ and $\frac{d(\cos\theta)}{d\theta} = -\sin\theta$.

The Analogy with the Hyperbolic Functions

We now look at the unit hyperbola $x^2 - y^2 = 1$ and define the hyperbolic functions in a manner which emphasizes the analogy between the circular and hyperbolic functions. Instead of a sector of a circle we consider the sector S subtended by the arc of the hyperbola from P(1,0) to R(x,y), $x > 0$, $y > 0$.

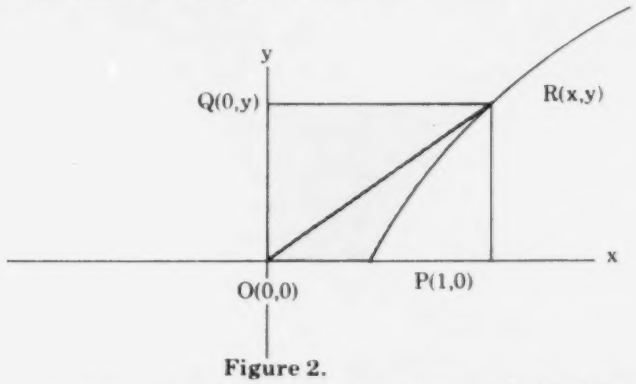


Figure 2.

Let $\frac{\theta}{2}$ be the area of the sector S as in the case of the circular functions. Next, in a manner analogous to the definition of the circular functions, we define $\cosh \theta = x$ and $\sinh \theta = y$. From this definition we obtain the analogous integral representation for $\theta = \sinh^{-1} y$ as follows. The area of the curved region OPRQ shaded in Figure 2 is

$$\int_0^y \sqrt{1 + t^2}\, dt.$$ Since the area of the triangle OQR is

$\frac{1}{2}\, OQ \cdot QR = \frac{1}{2}\, yx = \frac{1}{2}\, y\sqrt{y^2 + 1}$, it follows that

$$\int_0^y \sqrt{1 + t^2}\, dt = \frac{1}{2}\, y\sqrt{1 + y^2} + \frac{1}{2}\, \theta.$$

Following the same procedure as for the circular functions we can prove that

$$d \sinh \theta / d\theta = dy/d\theta = \sqrt{y^2 + 1} = x = \cosh \theta$$

and

$$d \cosh \theta / d\theta = y = \sinh \theta.$$

Next we obtain $\frac{d(x+y)}{d\theta} = dx/d\theta + dy/d\theta = y + x$ and therefore the sum,

$\cosh \theta + \sinh \theta$, has the defining property of an exponential function, i.e., $dE/d\theta = E$. We write $E(\theta) = \cosh \theta + \sinh \theta$ and note that $E(0) = 1$. Hence $E(\theta) = e^{\theta}$ (see Agnew, 1960). Moreover, since $\cosh^2 \theta - \sinh^2 \theta = 1$, it follows that $e^{-\theta} = \cosh \theta - \sinh \theta$. From this we obtain

(1) $\cosh \theta = \frac{e^{\theta} + e^{-\theta}}{2}$ and $\sinh \theta = \frac{e^{\theta} - e^{-\theta}}{2}$

Extension to the Complex Domain

We begin with the following elementary fact about complex numbers; every nonzero complex number u can be written uniquely as the sum of two complex numbers, $\chi(u)$ and $\psi(u)$, whose difference is the reciprocal of u, i.e., $u = \chi(u) + \psi(u)$ and $\frac{1}{u} = \chi(u) - \psi(u)$. Clearly χ and ψ exist and are unique if we define them as follows:

$$(2) \quad \chi(u) = \frac{u + 1/u}{2} \text{ and } \psi(u) = \frac{u - 1/u}{2}.$$

Furthermore, χ and ψ satisfy the equation $[\chi(u)]^2 - [\psi(u)]^2 = 1$. If u is positive and real it is easy to see that $\chi(u)$ and $\psi(u)$ are the hyperbolic functions cosh θ and sinh θ and $u = e^{\theta}$.

We turn now to the consideration of what happens when u is a complex number. There are two cases to consider:

1) u complex with modulus 1 and

2) u an arbitrary complex number.

We first show that we obtain the circular functions in Case 1.

In this case, since u lies on the unit circle, we set

$$(3) \quad u = x + i\,y \text{ and } 1/u = x - i\,y \text{ where } x^2 + y^2 = 1.$$

Then from our previous work we have $x = \cos\theta$ and $y = \sin\theta$ where

$$\theta = \int_0^y \frac{1}{\sqrt{1 + t^2}}\,dt. \text{ Thus, } \frac{du}{d\theta} = \frac{dx}{d\theta} + i\frac{dy}{d\theta} = -\sin\theta + i\cos\theta = i(\cos\theta + i\sin\theta) = i\,u.$$

Since $u = 1$ when $\theta = 0$ (or $y = 0$) it follows that $u = e^{i\theta}$ [See 2]. Therefore, from (2) and (3) we have $\cos\theta = x = \chi(u) = \frac{e^{i\theta} + e^{-i\theta}}{2}$ and $\sin\theta = y = \frac{1}{i}\psi(u) = \frac{e^{i\theta} - e^{-i\theta}}{2i}$.

Finally, for Case (2), we let u be an arbitrary complex number. In this case we may write $u = e^{x+iy}$ where $x = \ln|u|$ and $y = \arg u$, $0 \leq y < 2\pi$. If we let $z = x + iy$ it follows that $\chi(u) = 1/2(e^z + e^{-z})$ and $\psi(u) = 1/2(e^z - e^{-z})$, which define the complex functions of a complex argument, cosh and sinh. We simply note that $\cos z = \cosh iz$ and $\sin z = i \sinh iz$.

Closing Observations

In the preceding pages, we have obtained an almost completely analogous development of the circular and hyperbolic functions. There is, however, one aspect in which the analogy seems to break down. In the case of the circular functions the parameter θ may be regarded as arc length, but this apparently does not hold in the case of the hyperbolic functions. We claim, however, that one can make the analogy more complete by adopting a metric different from the usual Euclidean metric in the case of the hyperbolic functions. In fact, the appropriate metric in this case turns out to be the Lorentzian metric of the special theory of relativity (see Berman and Nomizu, 1984). In the Lorentzian metric, arc length is given by the formula

$$s = \int_0^{\theta} \sqrt{|(dx/dt)^2 - (dy/dt)^2|}\,dt$$

where the negative sign replaces the plus sign in the more familiar Euclidean metric. Substituting the parameter t for θ, we have $x = \cosh t$ and $y = \sinh t$.

Therefore $\frac{dx}{dt} = y$ and $\frac{dy}{dt} = x$ and hence

$$s = \int_0^{\theta} \sqrt{|y^2 - x^2|}\ dt = \theta.$$

Thus we obtained an interpretation of θ as arc length under a different metric in the case of the hyperbolic functions.

What we have shown is that if we begin with functions of a complex variable, $\chi(u) = 1/2(u + 1/u)$ and $\psi(u) = 1/2(u - 1/u)$, we obtain

$$x = \chi(u) = \cosh \theta = \frac{e^{\theta} + e^{\theta}}{2}$$

$$y = \psi(u) = \sinh \theta = \frac{e^{\theta} - e^{\theta}}{2}$$

for u real and analogously

$$x = \chi(u) = \cos \theta = \frac{e^{i\theta} + e^{i\theta}}{2}$$

$$y = -i\,\psi(u) = \sin \theta = \frac{e^{i\theta} - e^{i\theta}}{2i}$$

for u complex with modulus 1. In both cases θ may be interpreted either as area or as length of arc along the unit curve. Thus the analogy between the hyperbolic and circular functions is even more complete than is ordinarily thought.

REFERENCES

1. Agnew, R. P. (1960). *Differential equations* (2nd ed.). New York: McGraw Hill.
2. Berman, G. S. & Nomizu, K. (1984). Trigonometry in the Lorentzian geometry. *American Mathematical Monthly, 91*, 543-549.
3. Klein, F. (1932), *Elementary mathematics from an advanced standpoint.* New York: Macmillan.

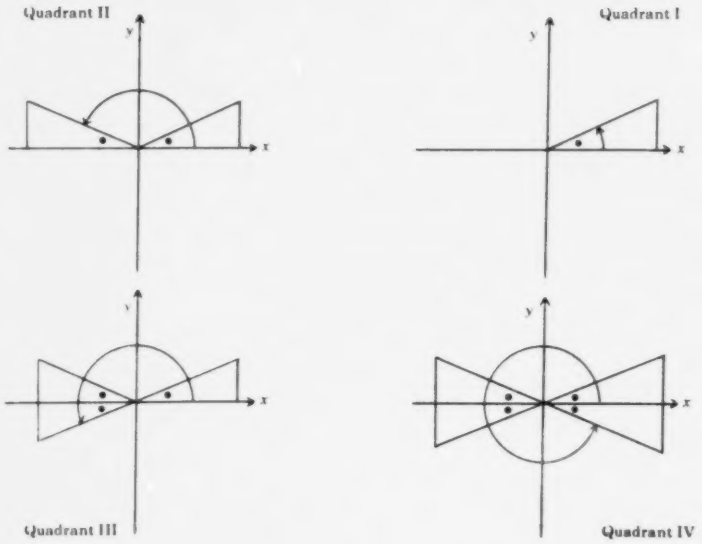
SHORT COMMUNICATIONS
"DON'T DESTROY THE BUTTERFLY"

By William Walworth
C.S. Mott Community College
Flint, MI 48502

Bill Walworth is a professor of mathematics and computer science at Mott Community College in Flint, Michigan. He received his master's degree in engineering from Yale University and has written numerous articles in a variety of educational journals over the past ten years.

One difficulty students encounter when attempting to determine a reference angle in trigonometry is to inadvertantly and incorrectly use the complement of the angles which exists between the terminal side and the y-axis. When I develop the idea of the reference angle in my classes, I tell my students: "Don't destroy the butterfly."

To show them that the reference angle for an angle in quadrants I-IV is always the angle between the x-axis and the terminal side, I proceed as follows:



As you can see, if they move outside the butterfly they will lose the reference angle.

It's a gimmick that adds a light touch to the class presentation and important "it works." The Japanese have had great success with gimmicks in their educational process. Maybe we should be trying more of them ourselves.

AN UNEXPECTED WAY TO AN EXPECTED VALUE

by Richard F. Maruszewski, Jr.
Lee Parsons
Mathematics Department
United States Naval Academy
Annapolis, MD 21402

USNA
U.S. Naval Academy

Lee and Dick are both Naval Reservists who have been recalled by the Navy to teach at the United States Naval Academy, Lee from the University of Southern Louisiana and Dick from Jamestown Community College in New York. Lee received his Ph.D. from the University of Wisconsin in Madison, and Dick received his from the University of Wisconsin in Milwaukee.

In a probability class, we illustrate the fact that the mean or expected value of a discrete but infinite probability distribution is $E(x) = \sum_{\text{all } x} x\,p(x)$ by an example such as the following:

Suppose that we are flipping a fair coin. Let the discrete random variable x be the flip on which the first head occurs. Then

$P(x = 1) = P(\text{Head}) = 1/2$

$P(x = 2) = P(\text{Tail and Head}) = (1/2)(1/2) = 1/4$

•

•

•

$P(x=n) = P(\text{Tail and Tail and } \ldots \text{ Head}) = (1/2)^{n-1}(1/2) = 1/2^n$

Therefore, $E(x) = \sum_{n=1}^{\infty} n(1/2)^n = 1/2 + 2/4 + 3/8 + \ldots .$

This series can easily be shown to converge by using the ratio test. The value that it converges to is usually obtained by showing that it is a special case of the derivative of a geometric power series. Consider instead the following geometric series:

$$\boxed{1/2} + \boxed{1/4} + \boxed{1/8} + \boxed{1/16} + 1/32 \ldots = (1/2) / (1-(1/2)) = 1$$

$$\boxed{1/4} + 1/8 + 1/16 + 1/32 \ldots = (1/4) / (1-(1/2)) = 1/2$$

$$\boxed{1/8} + 1/16 + 1/32 \ldots = (1/8) / (1-(1/2)) = 1/4$$

$$\boxed{1/16} + 1/32 \ldots = (1/16) / (1-(1/2)) = 1/8$$

$$\vdots \qquad\qquad \vdots$$

Now note that the sums of the circled numbers are the terms of the infinite series for E(x). Thus, because all the series are absolutely convergent, the value of E(x) is the infinite sum of the answers to the geometric series:

$$1 + (1/2) + (1/4) + \ldots = \frac{1}{1-1/2} = 2$$

Editor's Comment: In general, the discrete random variable X, where X is the number of trials until the first success occurs, has the expected value $E(X) = \sum_{n=1}^{\infty} n\theta(1-\theta)^{n-1} = \frac{1}{\theta}$; n is the number of trials and θ is the probability of success on each trial. In this article we have $E(X) = \sum_{n=1}^{\infty} n\frac{1}{2}(\frac{1}{2})^{n-1} = \frac{1}{1/2} = 2$. As a generalization, see *The AMATYC Review*, Spring/Summer, 1985, "Summing Infinite Series in Closed Form" by T. L. Boullion and Travis Thompson.

MATHEMATICS EDUCATION

QUESTIONBOOKS: Using Writing to Learn in Mathematics

by Agnes Azzolino
Mathematics Department
and
©1987, Robert G. Roth, Ed.D.
English Department

Middlesex County College
Edison, New Jersey 08818

Associate Professor of English at Middlesex County College, Robert G. Roth studied Mathematics at MIT before becoming an English teacher. He has published papers on the teaching of writing and was selected as a Finalist in the 1986 National Council of Teachers of English Promising Researcher Award Competition. As Writing Across the Curriculum Coordinator at his college, Dr. Roth has organized discussions and workshops among faculty in all subject areas.

Agnes Azzolino received her BA and MA from Montclair State College and has taught math at levels from preschool to adult school. She is at present working on a monograph entitled **How To Use Writing To Teach Mathematics.**

When students listen to our lectures or read texts we have assigned, most of it seems to make perfect sense to them. Why shouldn't it? The pieces all fit together and apt examples appear when needed, for we and the authors have struggled to get our ideas clear to ourselves and to find effective ways to present them to others. And during this struggling, since we are making connections, seeing relationships, and clarifying distinctions, we are learning more about whatever subject we are writing about. Unfortunately, those students complacently listening or reading are not learning nearly so much, for they are not struggling to organize their ideas. Because what they read or hear seems to make perfect sense to them, they believe they have understood it all. Asked to explain in their own words what they have understood, though, many would be left nearly speechless.

This need not be the case. If these students were asked to frequently compose as we have done, to write regularly about the subject matter of the course, most would learn much more. This is what the writing-across-the-curriculum movement is all about. Writing is not only a means of communicating what is already known, but is also a powerful learning tool. This is because writing combines many characteristics of successful learning processes. The student when writing is actively engaged in a

self-rhythmed procedure; he or she must make systematic connections and formulate them explicitly; and the result is a visible graphic product that presents the whole process for review, thereby facilitating reformulation and re-interpretation (Emig, 1977). Best of all, writing is the most readily available of all learning tools: the student needs only a pencil and a piece of paper.

Notice that the emphasis here is on the student's work, not on the teacher's. If students are to learn, they are the ones who must strive to get their ideas clear, who must search for the words that will structure and convey the relevant concepts. There are many practical ways to incorporate such experiences into a mathematics class. One is to provide short in-class writing assignments, asking students to explain a concept or describe a procedure. (See, for example, King, 1982, and Nahrgang & Petersen, 1986.) The logical extension of these in-class exercises is what we have christened the Questionbook, a carefully-structured, content-based course notebook kept by students throughout the semester.

In class, many teachers have tried asking students something like this: "Simplify the expression $(3x^3y)^2(-xy)$ and write your answer on a piece of paper where I can see it as I walk around the room." The next kind of question might be a more reflective one: "List the steps required to find the product of 145 and 23." Or, one we would love to have the time to ask in class: "How do the restrictions on a function and the domain of that function relate to each other?" Granted, we do at times ask questions like these in our lectures hoping that every student will personally, privately answer all of them. But there usually is not enough time to stop, permit the often necessary reflection time, and continue a smooth-flowing, coherent presentation. The Questionbook permits these questions to be answered, not just asked.

The Questionbook procedure is simple enough. At the beginning of the semester, students are given a list of questions based on the material covered in the course, most requiring that they compose fairly precise answers in clear English sentences. Generally, the questions ask students to perform one of five composing acts: write an accurate definition, describe a procedure, explain a concept, translate symbols into words, or create a word problem. (Sample questions within each of these five categories are presented in **Appendix A.**) In answering these questions, students are required to carefully revise preliminary drafts and to show all rough work. They are encouraged to hand in completed portions of the Questionbook regularly, and they are allowed to reformulate any answers deemed unacceptable until these problems have been mastered. When and if all questions have been acceptably completed, the student has earned the equivalent of an 'A' on one course test. (See **Appendix B** for more details on procedural matters.) This system motivates students while requiring that they do a great deal of reworking and rethinking, and thus learning.

Why use writing this way in a mathematics class? What does the Questionbook accomplish for students? The major benefit, of course, is the emphasis on conceptualization. The Questionbook problems help students get to the heart of the conceptual content, which is sometimes clouded by the algorithm. Students may know how to calculate the mean and median of a sample, for example, yet they will

not know what they are really doing when performing these calculations nearly as well as they will once they have figured out how to explain in what ways the mean and the median are alike and in what ways they are different. Similarly, a student who is able to correctly find the restrictions on a function and the domain of that function may all the time be confusing the meanings of those terms — since nearly identical procedures are involved in determining each. Thus, one of our students knew that, for the function $f(x) = 1/(x-3)$, $x \neq 3$ was the restriction, but when explaining how the restrictions on such a function and the domain of that function relate, she concluded that "The restrictions of a function and the domain are synonymous terms." That was one she had to rethink.

To compose appropriate answers to such questions students must make sense of the algorithm they have been given — explain it, justify it, or relate it to something else. In this way they are encouraged to make the meaning of the relevant algorithm explicit in their own words, to get the concept clear to themselves. The best students will do such thinking on their own initiative, but many will not — until the Questionbook is assigned. This fact explains the many practical benefits, as we shall see.

Review

The simplest of these benefits is review. Often the Questionbook provides students a way of re-experiencing a learning process they have had in class, thereby re-emphasizing and reinforcing the major lecture topics and course concepts. And the completed problems constitute extensive, clearly structured, and personalized review notes. (This is probably why the better students appreciate the Questionbook most just before the final exam.) It also helps students become more familiar with mathematical language by providing a context in which they will use appropriate vocabulary.

Memorization of terms is never the central goal though. The idea is to ask students to re-present, to present again, what they have learned, but now in a more generalized form. Asking them to explain in sentences how to graph a function or how to change a base 2 number to a base 10 number, for example, may prove more useful than asking them to graph five additional functions or to convert various base 2 numbers five more times. As educational psychologist Jerome Bruner (1970) says in "On Learning Mathematics," "our task as teachers is to lead students to develop concepts in order to make sense of the operations they have performed." To learn, Bruner explains, is "to climb on your own shoulders to be able to look down at what you have just done — and then to represent it to yourself." If we can in this way induce the re-evaluation of what has already been processed, both understanding and retention will be greatly enhanced.

Individualization

We have found that reviewing a student's Questionbook is an effective way of personalizing the teaching/learning process. Often the teacher quickly becomes familiar with the student's particular style and characteristic difficulties. Jotting down pertinent bits of strategic advice or even simple words of encouragement

becomes a way of carrying on an informal conversation with a student. In addition to C's (for "Correct"), R's (for "Redo it") and the usual summary comments like "Good," "Nice work," "No," and "Much better," Questionbooks rapidly become sprinkled with pertinent circles and squares, attention-directing arrows, and comments like "Close, but not exactly true," "Not always so," "How do you know this?" "Okay, this is an example, but what is it?" and "Can you make this part clearer?" In class, it is rarely possible to give this much personal attention to more than a few students, and often only the student who least needs such attention is confident enough to benefit from it in the public glare of the classroom.

Corrective Feedback

Some students may think they understand a term, procedure, or concept, but when they have to spell it all out they discover that they do not. Seeing where the gap is, they can then go back and figure it out for themselves — or else seek help, either from the teacher or from a fellow student. Even if they try to simply mimic what they have heard or read, they often find they cannot get it right. Having to write something out in words — whether it be a procedure, an analytic distinction, or a basic concept — is a way of checking to see that they have understood it in terms that make sense to them. Such an activity will often prompt them to ask themselves, "Well, just what did the teacher say? What did it mean?" For all of us, putting an idea into words means thinking it through for ourselves.

Similarly, when reviewing the Questionbook, the teacher has a means of discovering a student's — at times even a whole class's — misconceptions. Because students must make their understanding explicit, the teacher is in a much better position to identify the sources of confusion. And learning through this process is facilitated by requiring students to rework inadequately answered questions until mastery is achieved.

Transferability of Learning

The ultimate test of any teaching strategy is transferability: Can the students take what they have learned and transfer it to new contexts, apply it in unfamiliar situations? Writing as carried out in the Questionbook increases transferability. It does so because it requires students to make the concrete abstract — that is, to deliberately name and systematically organize the specific features of what they have learned. Students may be temporarily able to follow an algorithm, to apply a particular formula or carry out a specific procedure. However, unless they think through just what each step represents and why the action is performed in this way rather than in that, they are unlikely to even retain the algorithm for long, never mind understand its underlying form. Knowledge developed and reinforced in this more deliberate manner is most likely to be transferable to new contexts. Students will be better able to recognize the unexpected applicability of the algorithm if they have had to both act it through and think it through.

Today, many teachers of mathematics are recognizing the value of using writing-for-learning to encourage students to "think it through" in this way. We believe that the Questionbook embodies a particularly apt and effective method. We know it is

versatile, having used it to advantage in both advanced and remedial classes. Because it is a more focused, content-directed approach than the open-ended journal in which students are asked to invent analogies or discuss their difficulties, it provides students with a better means of reviewing course content. They can even use their Questionbooks for a refresher course months or even years later. In any case, this is a writing-to-learn method most teachers of mathematics are likely to find congenial.

References

1. Bruner, J. S. (1970). *On knowing: Essays for the left hand.* New York: Atheneum.
2. Emig, J. (1977). Writing as a mode of learning. *College Composition and Communication, 28*(2), 122-128.
3. King, B. (1982). Using writing in the mathematics class: Theory and practice. In *Teaching writing in all disciplines.* San Francisco: Jossey-Bass.
4. Nahrgang, C. L. & Petersen, B. T. (1986). Using writing to learn mathematics. *Mathematics Teacher, 79*(6), 461-465.

APPENDIX A: SAMPLE QUESTIONS

I. Write a Definition:
   1. Complete: **Absolute value** means _____ .
   2. What's an **irrational number?** (State your answer in terms of a fraction.)
   3. State: a) a sloppy definition of **slope;**
      b) a clean definition of **slope.**
   4. State, in your own words and as briefly as possible, the characteristics of a **binomial random variable.**
   5. Describe the graph of $y = b^x$, where b is a positive integer. (Include the words **asymptote, x-intercept,** and **y-intercept** in your description.)

II. Describe a Procedure:
   1. How do you change a base 2 number to a base 10 number?
   2. Explain one way of figuring out the prime factorization of 132.
   3. a) Multiply: $(x + 5)(x - 3)$;
      b) Explain, in 4 or fewer steps, how to complete the computation mentally and write just the product.
   4. How do you graph an equation like $y = 4x + 3$?
   5. How do you graph a function?
   6. How do you change a fraction to a decimal? (Skip fractions that may be changed mentally, that have denominators easily changed to a power of 10, or that you've memorized because of frequent use. Answer the question in general.)

III. Explain a Concept:

1. How do the restrictions on a function and the domain of that function relate to each other?
2. Prove that -3/4 is the reciprocal of -4/3.
3. In maximizing or minimizing a function over an interval, which values of the function must be computed and why?
4. Why does $\lim_{x \to 2} \sqrt{x}$ exist but $\lim_{x \to 2} \sqrt{x-4}$ not exist?
5. State the relationships that exist between velocity and acceleration.
6. What's the difference between ratio and proportion?
7. What does reliability have to do with statistical inference?

IV. Translate:

1. Translate into words: If f and g are functions,
$$(f \cdot g)' (x) = f'(x) \cdot g(x) + g'(x) \cdot f(x).$$
2. Translate into words this equation for use with trapezoids:
$$A = (1/2)(h)(a + b).$$
3. Complete: The symbol P(E) is read _____.
4. Translate into symbols: The derivative of a constant times a function is the constant times the derivative of that function.

V. Create a Word Problem:

1. Write a word problem which requires this computation in order to find the solution $658:
   a) 9 x $167 = $1503
   b) $450 + $273 + $1438 = $2161
   c) $2161 - $1503 = $658.
2. Make up a question that requires the computation of $C_{10, 4}$ to answer the question. Also, complete the computation.
3. Make up a question that requires the computation of $P_{10, 4}$ to answer the question. Also, complete the computation.
4. Write a word problem that requires calculation of the area of a figure composed of at least 3 basic figures (squares, rectangles, triangles) and whose total area is exactly 72 square centimeters.

APPENDIX B: ORGANIZING THE QUESTIONBOOK

On the first day of class we distribute to each student a complete set of questions, the Direction Sheet, and the Cover/Grading Sheet. (See sample Sheets below.) We discuss the stages (parts a-d) each response must go through and the possible grades each question might receive. Students are advised not to answer questions before the topic has been presented in lecture.

Once lectures have begun, a due date is set for collecting a few questions to get the process started. After this, our practice is to accept questions whenever a student is ready to hand them in, though it becomes necessary to establish deadlines too. We usually return the graded questions as soon as possible so students will feel they have accomplished something.

The Cover/Grading Sheet makes grading a manageable job. It should have blanks long enough for the multiple grading of each question. Blank space provides room for making overall comments; hints, leading questions, and suggestions may be written in the Questionbook proper. Frequent deadlines (e.g., Last Day for First Grading of Questions 1-20) will facilitate grading and question completion. We also set a date for the Last Grading Before the Final Due Date so that students will have one last opportunity to receive our input before completion of their Questionbooks. A date is chosen early enough to allow the instructor time to complete the grading and return questions to students for their final revisions. At this time numbers of all missing or inadequately-answered questions are listed, as on the sample sheet below. Graded Questionbooks are returned to students at the last regular class meeting for use in preparing for final exams.

DIRECTION SHEET
MAT-000 COURSE QUESTIONBOOK
for A. Azzolino

This assignment exists to:

1. help the student assimilate and organize information;
2. determine if the student has acquired correct and complete information;
3. help student to determine important points of lecture and course content;
4. improve the student's overall grade.

The assignment consists of the answering of four-step questions **and their collection in notebook form.** It is worth one test grade.

To obtain a grade of A, a student must receive the mark of OK or C for correct for each questionbook question. All other grades will be awarded at instructor discretion. Questions may be graded repeatedly without penalty. Possible grades on each question are:

C or OK - for CORRECT, full credit, DONE. Put this with your completed questions for collection at course end;

I - INCOMPLETE, redo or complete for full credit. You must still work on this one;

R - Redo, needs correction, redo for credit. You must still work on this one.

QUESTIONS MAY BE GRADED REPEATEDLY WITHOUT PENALTY UNTIL THE FINAL DEADLINE!!

Each question must be answered in the following manner.

1. Questions are to be answered as if you were speaking or writing for a fellow student or for your own future use.

2. ANSWER EACH QUESTION ON A SEPARATE SHEET OF PAPER. WRITE ON ONE SIDE OF THE PAPER! Each page must have your last name and the question number as a heading.

3. Each question has the following parts:

Part a. THE QUESTION - assigned question must be copied in your hand.

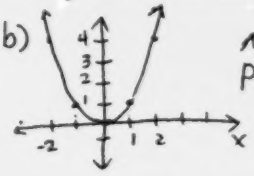
Part b. SCRATCH WORK - written in your own hand. your thoughts about the question: scratch work.

Part c. ROUGH DRAFT - write your answer, read your answer, REVISE your answer, perhaps have someone else read it. It should be in complete sentences, etc. (It must be in your own hand.)

Part d. FINAL ANSWER - The cleaned-up or rewritten copy of part c. (It must be in your own hand.)

Your Last Name
1.9.0

1.9.0a) What does the graph of $y = x^2$ look like?

b) 

vertex at (0,0)
parabola like ∪
concave ↑

c) $y = x^2$ is a ~~parabola~~ parabola with a vertex at (0,0). It opens up and looks like a ∪. When x is small, y is large.

d) $y = x^2$ is a parabola. The graph has a vertex at (0,0) and looks like an ever-widening ∪. As x increases, y first decreases to zero then increases. The curve is symmetric.

Mat000 QUESTION BOOK COVER/GRADING SHEET
Student's Name

SUBMIT ONLY THOSE QUESTIONS WHICH NEED TO BE GRADED!

HOLD ALL GRADED, CORRECTED, QUESTIONS IN A SAFE PLACE.

16
17
18
21
26
36

8/20
Don't answer quest.
until after lecture!
cf²

1. C
2. C
3. C
4. R C
5. C
6. C
7. C
8. C
9. C
10. R R C
11. C
12. R I C
13. R ~~o~~C
14. C
15. C
16. I see me
17. R
18. R
19. C
20. C
21. R
22. C

16, 17, 18, 21

23. ~~I~~ C
24. C
25. C
26. R
27. C
28. C
29. C
30. C
31. R R ~~see me~~ C
32. C
33. C
34. C
35. C
36. I
37. C
38. C
39. C
40. C good!
41. C
42. C
43. C

26, 36

WHAT DIFFERENTIAL EQUATIONS BOOKS LACK

By George E. Abdo
Florida Institute of Technology
Melbourne, FL 32901

George received his Ph.D. in mathematics from Texas A&M University in 1969 and has taught mathematics at Florida Institute of Technology since his graduation. He received a "Teacher of the Year" award for the 1972-73 academic year.

A student of mine once told me

It's easy to solve the equation

$$xy''(x) + y'(x) + xy(x) = 0.$$

Just write down the characteristic equation . . . Let's see, you replace y'' by m^2, y' by m, and you drop the y so to get

$$xm^2 + m + x = 0.$$

Now, oh yes, you solve for m:

$$m = \frac{-1 \pm \sqrt{1 - 4x^2}}{2x}.$$

So the solution is

$$y = c_1 e^{\left(\frac{-1 + \sqrt{1-4x^2}}{2x}\right)x} + c_2 e^{\left(\frac{-1 - \sqrt{1-4x^2}}{2x}\right)x}.$$

This is a typical mistake students make in differential equations and I partly blame textbooks in this subject. Let me explain:

In virtually every calculus textbook there is a miscellaneous section at the end of the chapter on integration methods, whereby a collection of integrals is given in random order so that the student will have a chance to decide which method is applicable to evaluate each integral. In like manner, a collection of differential equations is needed at the end of a differential equations text for the same purpose. Equations that have not been discussed in the text should also be included since it is important that the student knows which ones he should be able to handle and which ones he cannot. No hints such as using t for the independent variable for equations

suitable for Laplace transforms should be given. A small sample of such a list might look like this:

1. $x^2y'' + y = 0$
2. $y'' + x^2y = 0$
3. $y'' + y = x^2$
4. $y'' + y = 1/x^2$
5. $x^3y'' + y = 0$
6. $y'' + y = h(x-1) - h(x-1)$, where $h(x)$ is the unit step function
7. $y'' + \sin y = 0$

Look at the first four examples. They differ only by the position of x^2 and yet they are very different. The first one is the Euler equation and its solution is relatively simple. Infinite series is the obvious choice for the second one. Undetermined coefficients, Laplace transforms, and even variation of parameters or series could be used in the third one, although I would stay away from the last two. Some students may argue that since no initial conditions are given, Laplace transforms could not be used, "... after all, all differential equations in the Laplace transform chapter have initial conditions and if that were not enough, they all have t (for time) as their independent variable." The fourth one's preferred method is variation of parameters. The fifth one has an irregular singular point at the origin and therefore is not solvable about the origin by elementary methods. I'd vote for Laplace transforms to solve the sixth one. The seventh one is of course non-linear. The only hope is to reduce it to a first order equation since the independent variable x is missing. This does not guarantee the solution of the first order equation.

A second type of problem that should be included at the end of the book is the equation that can be solved by various methods. For example, ask the student to solve

$$y'' + y = \sin x$$

by the methods of

a) undetermined coefficients
b) variation of parameters
c) series
d) Laplace transforms

and show that all answers are equivalent. This helps to consolidate ideas, and to unify and contrast methods.

The collection of all these problems could comprise the last chapter of the book. The solution of a selection of them could be used as a review of the course.

(Editor's Note: The author personally told me that he is finding that many adjunct faculty who lack teaching experience are being assigned to teach differential equation courses. Therefore, he feels that it is essential that textbook authors make their books pedagogically sound.)

Proposals Sought for the 1989 NSF/CBMS Regional Conference Series in Mathematical Sciences

PROGRAM The National Science Foundation is seeking proposals from host institutions in the U.S. for five-day regional conferences. Each conference is to feature a distinguished lecturer who will deliver 10 lectures on a subject of current research interest in the mathematical sciences. Each lecturer agrees to prepare a monograph within six months based on his or her lecture notes. It is expected that the lecturer will give two lectures per day during the five days, with the remaining time available for study, informal discussion, and exchange of ideas.

CRITERIA Institutions having at least some research competence in the area of its proposal are eligible and encouraged to apply. The conference should be planned for a summer week in 1989 (not earlier than May 1) or held during a recess in the succeeding academic year.

CONFERENCE ORGANIZATION The host institution is responsible for arranging conference details, hotels, and social events during the five-day conference. Participants in the conference will receive allowances for travel and subsistence under the host institution's grant from NSF.

AWARD The principal lecturer receives from the Conference Board a fee for delivering his or her lectures and a second fee upon approval by the NSF of the substantial expository monograph derived from the lectures. These monographs are published in various series by one of three organizations: the American Mathematical Society, the Society for Industrial and Applied Mathematics, or, in conjunction, the American Statistical Association and the Institute of Mathematical Statistics.

APPLICATION PROCEDURES Proposals (twenty copies of each) by prospective institutions should be sent directly to the Data Support Service Section, National Science Foundation, 1800 G Street, NW, Washington, DC 20550. Proposals must follow the guidelines issued by CBMS.

FOR INFORMATION SHEET AND GUIDELINES WRITE TO:

Conference Board of the Mathematical Sciences
1529 Eighteenth Street, NW
Washington, DC 20036
202-293-1170

Deadline for 1989 Conferences: April 1, 1988

REGULAR FEATURES

EDUCATIONAL REFLECTIONS

FUTURE SHOCK: HAND HELD COMPUTERS

by Gregory D. Foley
Ohio State University
Columbus, OH 43210

Gregory D. Foley has been Assistant Professor of Mathematics Education at Ohio State University since 1986, when he received his Ph.D. from the University of Texas at Austin. Prior to that he taught mathematics for 4 years at North Harris County College, Houston, Texas, and for 5 years at Austin Community College, Austin, Texas. He has been active in AMATYC for several years and is currently chairing the Qualifications Subcommittee of the Education Committee. His research interests include calculus and precalculus education, the transition from high school to collegiate mathematics, curriculum, hand-held computers, and the history of mathematics.

Most of us still remember slide rules; many of us know how to use them, but few of us still do. Slide rules have become "old tech." Computers used to fill rooms, but micros have become commonplace. Now there is a new generation of computer — the hand-held computer. These new machines are calculator-sized, inexpensive, and can do things that some micros can't. Beamer's lament, "It is somewhat unsettling to have a $60 calculator appear to outperform a multi-thousand-dollar computer" (1987, p. 155), may be prophecy. Prepare for the shock!

Calculators and microcomputers have been causing many mathematics educators to rethink the priorities, content, and pedagogy of school mathematics for some time. The *Agenda for Action* (National Council of Teachers of Mathematics, 1980) stated: "Mathematics programs must take full advantage of the power of calculators and computers at all grade levels" (p. 8). This rethinking came to a head at the collegiate level in January 1986, when the Sloan Foundation sponsored a 4-day conference on calculus instruction at Tulane University in New Orleans. The group of 25 administrators, mathematicians, scientists, and educators that met at Tulane emerged with recommendations for a new calculus course that focuses on central topics and utilizes symbolic-manipulation and interactive-graphics computer systems (see Tucker, et al., 1986, especially sections 3 & 7). Now such computer systems are built into hand-held devices such as the Casio fx-7000G, the Hewlett

Packard HP-28C, and the Sharp EL-5200. All three possess on-screen programming and interactive graphics. The Hewlett-Packard has a computer algebra system, which can perform many of the operations of calculus and linear algebra. They are not just calculators; they are computers.

The Issues

The presence and affordability of these new machines raise some issues that we all need to consider: How will these new devices affect mathematics education at the two-year college? How *should* they affect it? How should they influence what we teach and how we teach it? How can they be used to improve our students' understanding of mathematics?

Small, Hosack, and Lane (1986) raised still other, more specific questions concerning the role of computer algebra systems in undergraduate mathematics instruction. I wish to address one of their questions relative to the Casio fx-7000G: "Should a student analyze a function to sketch the graph, or just call up a graphing program?" (p. 433). I choose the Casio as my focus because I have been teaching a precalculus class at a local high school in which each student has been issued a Casio, as a part of the Ohio State University Calculator and Computer Precalculus Project. Tucker (1987) described some of the capabilities of the fx-7000G and the HP-28C, giving much greater attention to the more powerful and more expensive Hewlett-Packard machine with its MACSYMA-like capabilities. So, I shall begin my response to Small, et al., by describing the fx-7000G:

This machine combines the capabilities of a scientific calculator, a programmable computer, and an interactive-graphics computer system. The display window can display a 16-character by 8-line set of textual symbols, and graphs are produced on a 95 x 63 dot matrix. Commands and programs can be selectively changed and reexecuted. The Casio uses true algebraic logic, which parallels standard mathematical notation. Any elementary function can readily and rapidly be graphed. Some simple BASIC-like programming is required to graph conics, polar equations, and parametric equations, and the point-by-point plotting for polar and parametric equations is quite slow, typically taking about 60 sec. But it can be programmed to execute virtually any two-dimensional graph. For example, the first 35 microcomputer-based graphs in Montaner (1987) can be replicated on this wallet-sized device. Each unit comes with an owner's manual that describes its operation and includes a library of 14 programs.

It should be noted, however, that in order to be able to take advantage of the problem-solving power of this machine, students must first be able to read and interpret graphs. As the Second International Mathematics Study (SIMS) indicates, many students are weak in graphing. SIMS achievement data show little knowledge or growth among United States precalculus 12th graders on the topics of coordinate geometry, functions, and graphing (Travers, 1985). In particular, SIMS data indicate most U.S. 12th graders cannot make a strong connection between a function

and its graph. For instance, most do not realize that the solution for a system of two equations corresponds to the intersection points of their graphs. Although the graphics calculator can help students develop the graph-equation connection as well as other basic understandings about graphs, experience indicates that we must be prepared to reach back and teach some fundamental concepts and skills before students can effectively use such machines.

Using the Pocket Grapher

Students first must learn the keyboard, which is substantially more complicated than that of a standard scientific calculator. Then it is a simple matter to graph most elementary functions. Because the stadard Casio "viewing rectangle" is [-4.7, 4.7] x [-3.1, 3.1], in order to view the graphs of many functions the viewing rectangle must be altered. Both the viewing rectangle and the scale are set by using the **Range** feature. The next step is to learn how to locate points on the graph using **Trace**: this feature allows pixel-to-pixel movement along the most recently drawn graph, with the computer displaying the x- or y-coordinate associated with each pixel along the way. It is easy to stop, reverse direction, and switch from x-read-out to y or vice versa. The **Factor** command permits zooming-in or zooming-out about a plotted or traced-to point. The zooming factor can be chosen to be different for x and y if desired. It takes a while for students to become proficient at each of these three operations.

Once these operations are mastered, students can graphically solve equations and inequalities as well as systems of such by zooming-in. These operations also open the door to exploring local and global behavior, solving extreme-value problems, determining intervals over which a function is increasing or decreasing, and solving a host of other types of problems. The same methods apply regardless of the functions involved: $six\ x = \ln x$ can be solved as readily as $2x = 6$. Typical calculus problems, such as:

> An open box is to be made from a rectangular piece of sheet metal 20 cm long and 15 cm wide by cutting and removing equal square pieces from each corner of the rectangular sheet and then bending up the sides. What are the dimensions of the box with the largest volume that can be so constructed? What is the maximum volume?

become precalculus problems. Students must still understand the problem and be able to set up a function to express volume in terms of the common side length of the deleted squares. In addition, they must be able to choose an appropriate viewing rectangle and scale using **Range**, to key in the function, to read and interpret the graph, and to use the **Trace** feature in order to locate the coordinates of the maximum point. The graph of the defining equation for the volume function, $y = x(15 - 2x)(20 - 2x)$ is shown in Figure 1 as it would appear on the Casio screen using a viewing rectangle of [-1, 8] x [-50, 500], an x-scale of 1, and a y-scale of 50. Tracing to the maximum point then yields a read out of $y = 379.0377662$ (see Figure 2), and then with the push of one button, $x = 2.829787234$.

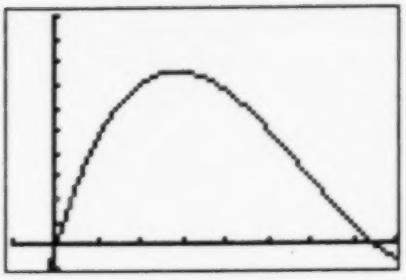


Figure 1

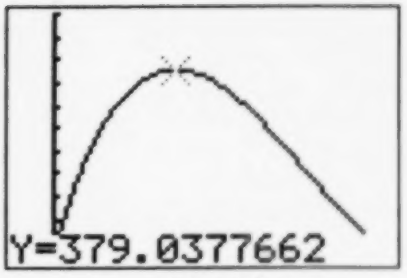


Figure 2

Figure Captions

Figure 1. The graph of $y = x(15 - 2x)(20 - 2x)$ as it would appear on the screen of the Casio fx-7000G using a viewing rectangle of $[-1, 8] \times [-50, 500]$, an x-scale of 1, and a y-scale of 50.

Figure 2. The graph of $y = x(15 - 2x)(20 - 2x)$ together with the read out of the relative maximum value of y over the interval $0 < x < 7.5$ as it would appear on the screen of the Casio fx-7000G.

Here x represents the height in cm of the box and y, the volume in cm^3. The other dimensions would be 9.34 cm and 14.34 cm. By the way, the standard calculus solution would yield $x = (35 - \sqrt{325})/6 \approx 2.828707270$ and $y \approx 379.0378082$. This 10-place accuracy could be achieved, if desired, by using **Factor** to zoom-in on the maximum point.

Conclusions

Should a student analyze a function in order to sketch the graph or vice versa? We have seen that — as Small, et al. (1986) implied — the student can now sketch the graph in order to analyze the function. Hand-held computers, however, do not give us mathematics made easy. Nevertheless, precalculus instruction that integrates the use of interactive graphics does have certain inherent advantages over traditional instruction:

1. The speed of the computer allows students to study enough examples of the graphs of functions for them to generalize properties about classes of functions; for example, that all quadratic functions graph as parabolas.

2. Typical textbook exercises are often contrived so as to be tractable using conventional paper-and-pencil algorithms. The computer enables students to solve realistic problems the mechanics of which would be too demanding or impossible if limited to paper-and-pencil techniques alone.

3. The graphics algorithms that a pocket computer permits are new and far-reaching. They allow students to solve large classes of problems by a single method. For example, equations involving any type of elementary function can be solved by the zoom-in method; whereas their solutions require a myriad of paper-and-pencil methods.

4. A graphics machine permits the student to obtain several views of a graph, to superimpose several graphs within the same reference frame, and to experiment with the scaling of axes — all without tedious number crunching and without the need for specialized (artistic?) motor skills.

Interactive graphics is now available on affordable, calculator-sized machines. Students will buy and use these devices whether or not we do. Although the graphics of a pocket computer is accessible to typical precalculus students, most of them need guidance to take full advantage of such graphics. As educators we must withstand the shock of yet another technological innovation so that we can provide help and direction to our students.

References

1. Beamer, J. E. (1987). Lessons learned while approximating pi. *Mathematics Teacher, 80,* 154-159.
2. Montaner, F.R. (1987). Use of the zoom in the analysis of a curve. *Mathematics Teacher, 80,* 19-28.
3. National Council of Teachers of Mathematics. (1980). *An agenda for action: Recommendations for school mathematics of the 1980s.* Reston, VA: Author.
4. Small, D., Hosack, J., & Lane, K. (1986). Computer algebra systems in undergraduate instruction. *The College Mathematics Journal, 17,* 423-433.
5. Travers, K. J. (Director). (1985). *Second international mathematics study: Detailed national report, United States.* Champaign, IL: Stripes Publishing
6. Tucker, T. (1987, January-February). Calculators with a college education? *Focus: The Newsletter of the Mathematical Association of America,* pp. 1, 5.
7. Tucker, T. (Chair), Goldstein, J., Lax, P., Raphael, L., Rodi, S., van der Vaart, R., Van Valkenburg, M.E., & Zorn, P. (1986). Calculus syllabi: Report of the Content Workshop. In R. G. Douglas (Ed.), *Toward a lean and lively calculus: Report of the conference/workshop to develop curriculum and teaching methods for calculus at the college level* [MAA Notes No. 6] (pp. vii-xiv). Washington, DC: The Mathematical Association of America.

PANDORA'S RECTANGULAR PARALLELIPIPED

by $(28867 \times 11 \int_{\beta}^{NER} SEC^2X \quad dX)$

In what threatens to be a regular feature in *The AMATYC Review*, this column will attempt to present the best (and sometimes the worst) anecdotes, puns, cartoons, jokes, and other nonsensical-but-fun things having something to do with the great subject of mathematics.

Some examples come to mind.

—— What every high school sophomore wants for Christmas:
GEOAMETRY [An "A" in geometry]

—— A hidden word: $\int \frac{1}{\text{cabin}}$ d cabin [= log cabin + c
= log cabin + "sea"
= "houseboat"]

—— A fractured saying: ENTURY ["Long time, no see".]

—— Finagler's Constant: That number which, when added to your answer, results in the author's answer.

—— A suggested team name for mathletes at a Texas university:
THE HOUSTON EULERS

—— A book worth skimming: **A COMPLETE TABLE OF EVEN PRIMES.**

—— A tricky question, a trickier response: A sneaky instructor included this problem on the final exam for a number theory course: "Prove that, if n is an integer larger than 2, there are no positive integral solutions X, Y, Z for the equation

$$X^n = Y^n = Z^n .$$"

One enterprising student wrote on the edge of the page, "I have found a proof, but the margin is too small to contain it."

—— An intriguing (?) sentence: "The tetrahedron divides all of Universe into all the tetrahedral nothingness of all the cosmic outsideness and all the tetrahedral nothingness of all the cosmic insideness of any structurally conceived or sensorially experienced, special case, uniquely considered, four-starry-vertex-constellared somethingess of human experience, cognition, or thinkability." (R. Buckminster Fuller, **SYNERGETICS 2**, [Macmillan, 1979]; page 213.)

Readers are invited to submit appropriate items. Citations of sources must be included for non-original material. Cartoons and other drawings must be camera ready. The only reward offered is to put your name in print. (In certain cases, a better reward may be to **not** print your name.)

An initial contest is hereby proposed. Invent a new "metric" unit. The least-awful entries will be published in a subsequent column. For example:

PROF: The fundamental unit of hot air. One prof is the quantity of hot air generated in a typical 50-minute calculus lecture in early September. It is estimated that a kiloprof-strength hurricane would detach Florida from the mainland.

Contributions to this column should be sent to:

L. R. Tanner
Mathematics Department
The College of Idaho
2112 Cleveland Blvd.
Caldwell, ID 83605

BOOK REVIEWS

Edited by John J. Edgell, Jr.

FORTRAN 77 A TOP-DOWN APPROACH, Nonna Kliss Lehmkuhl, Macmillan Publishing Company, Inc., New York, 1983, 462pp, ISBN 02-369390.

Although this book is well written and contains numerous examples, it does not have any distinguishing features which set it apart from the Fortran 77 books currently available. Therefore, if you are looking for something new and innovative, look elsewhere. However, if you want a book with clear and complete presentations which can be used for a one semester course in Fortran 77, this book will do fine.

The book provides a gradual introduction into the programming process which makes it good for beginners. If you like flow charts, you will find them throughout the book. Twelve comprehensive chapters and an extensive appendix provide all that is needed to learn Fortran 77. All loop control structures are presented in typical fashion with nothing left to the imagination. Two chapters, as one might expect, are used to cover arrays (two and three dimensional). The chapter on formatting is solid but could have contained more examples while the chapter on file processing is a bit weak.

You really can't go wrong with this book, but if you are looking for the "something else," keep looking.

Reviewed by Louis Gioia, Mathematics Department, Nassau Community College, Garden City, New York 11530.

TOPICS IN CONTEMPORARY MATHEMATICS, Third Edition, Jack R. Britton & Ignacio Bello, Harper & Row Publishers, New York., 1984, 628pp., ISBN 0-06-040922-4.

This book is an excellent choice for a Liberal Arts Math course. It contains a varied selection of topics ranging from Logic and Sets to Geometry. Therefore, it should be very easy selecting a course outline from the table of contents.

There are over 3000 exercises included in this edition. Many of these were not in either of the two earlier editions. The chapter on consumer mathematics has some particularly good examples which should not be missed. Also, your students might find the chapter on calculators and computers most informative.

You should have no trouble following the order of topics the authors provide. It is quite logical and natural. Topics which can be skipped while still preserving the

established order are identified. There is an Instructors Manual which includes the usual teaching aids such as tests, answers to selected exercises, chapter suggestions, etc.

In conclusion, I would not hesitate to use this book for my class and I would recommend you try it.

Reviewed by Louis Gioia, Mathematics Department, Nassau Community College, Garden City, New York 11530.

VIDEOTAPES TO ACCOMPANY UNDERSTANDING ELEMENTARY ALGEBRA, Arthur Goodman and Lewis Hirsch, West Publishing Company, St. Paul, Minnesota 55102. (Complimentary videotapes with order of **Understanding Elementary Algebra,** Goodman and Hirsch, 1986, 460pp. ISBN 0-314-93532-0.)

The three videotape cassettes included in this package cover each chapter of the Goodman/Hirsch textbook in ten segments. Each topic is given a short introduction and then a series of problem examples illustrating simple to complex variations. The explanations and examples reflect those of the text itself. Each segment is approximately an hour in length, although a few run a little shorter. An instructor's guide is included containing all of the examples worked out on the tapes, lists of related examples in the textbook and appropriate homework exercises in the text for each problem type.

The lessons are demonstrations by an instructor at a chalkboard in a classroom without students. The individual topics are not developed but only reviewed on the assumption that the student has already been exposed to the material in class and text. The lack of a student audience is a drawback in the programs. At times it gives the impression of the instructor teaching to the camera. Typical student questions at difficult points could have made the presentations more interesting and relevant to the student audience.

The tapes are, for the most part, well done. General rules are usually written on the board, problems are explained slowly and in detail and important cautions are noted. This is true in all the tapes except the first on the Integers, which is crude and distracting. Rules for signed numbers, for example, are not written on the board, the camera sometimes faces a blank board, the chalk breaks, and the instructor pulls her microphone off. This certainly lends an air of human frailty to an otherwise dry subject but these distractions should have been edited out. Later segments are noticeably more polished. An additional suggestion would be for one segment to be devoted entirely to verbal problems, a traditional source of difficulty for most students.

These videotapes are generally suitable for student personal review and reinforcement. They would be especially useful for the student who has missed class

and would like to see a "live" presentation to supplement the reading and examples in the text.

Reviewed by Catherine Folio, Department of Mathematics, Brookdale Community College, Lincroft, New Jersey 07738.

ESSENTIALS OF TRIGONOMETRY, 2d ed., Karl J. Smith, Brooks/Cole Publishing Co., Monterey, California 93940, 1986, 292pp. ISBN 0-534-05274-6.

Amid all of the combined algebra and trigonometry texts that come across my desk, a book concentrating solely on trigonometry is a relative rarity. **Essentials of Trigonometry** by Karl J. Smith is such a book, and a fine one at that.

Smith follows the course of first discussing the trigonometry of the right triangle. Upon this foundation are built the definitions of the trigonometric functions of all angles and eventually of all real numbers. In addition to chapters devoted to the other standard topics of trigonometric identities (two chapters), trigonometric equations, and oblique triangles, are chapters devoted to complex numbers, polar coordinates, and logarithms. The overall presentation is mathematically sound and highly readable.

A particularly nice feature is the special sections devoted to real-world applications that accompany each chapter on distinctly colored pages. These applications are truly interesting. More standard applications are woven throughout the text and exercise sets. Also included throughout are instructions on how to use a calculator in conjunction with most topics. Distinctions between exact answers and approximate answers are clearly made.

The physical layout of the book is fine. Color is used to high-light key terms and set off formulas and definitions. Chapter summaries include important terms and questions keyed to specific objectives and sections within the chapter. Suggestions for cumulative reviews are also provided.

For any teacher seeking a textbook for a one-term trigonometry course, this book is worth looking at.

Reviewed by Deborah R. Levine, Mathematics Department, Nassau Community College, Garden City, New York 11530.

Send reviews to: Dr. John J. Edgell, Jr.
Editor, Book Reviews
The AMATYC Review
Mathematics Department
Southwest Texas State University
San Marcos, Texas 78666

THE PROBLEM SECTION

Edited by Michael W. Ecker, Ph.D.
Mathematics Department
Pennsylvania State University
Wilkes-Barre Campus
Lehman, PA 18627

Welcome to another Problem Section!

Activity has been pretty light, with not so many solutions lately. I'm savvy enough to know that the problems may have gotten too hard. Sorry about that folks! Let's see whether we can't bring it down to a more manageable level.

I have fewer proposals than I'd like to have on hand, but thanks to a few people, I have enough to eke by for another half a year. Thanks to those participating. And the rest of you: please let me hear from you!

It was nice seeing many of you at our convention in San Francisco. Prof. Steven Blasberg of West Valley College, for instance, handed me a couple of nice problems when we met there, both of which should appear this issue.

Prof. Joseph Browne of Onondaga Community College sent me some items. Joe is a fellow problem editor of long standing and does similar chores as I for *Mathematics and Computer Education*, another two-year-oriented publication. In such capacity, he points out that our problem K-2 (dealing with $a^b = c$) is like that publication's #219 in the Spring 1985 and Spring 1986 issues. I asked about combinations of a, b, c being rational and irrational, or 8 questions. The MACE problem, submitted by Gene Zirkel of Nassau Community College, was more ambitious, allowing complex numbers, thus admitting 27 possibilities. (All turn out to be possible, according to Browne.) Oh well: great minds converging upon common themes . . .

For those of you computerizing, I welcome computer-related submissions, but bear in mind that computer activity here should supplement, not supplant, the mathematics. In fact, I have nine computers now, my latest being an IBM AT clone, PC's Limited 286, an 8 mH machine with 21 meg hard disk and loads of goodies. Once you have a hard disk, well, you're spoiled for life. I also have a Franklin Ace 500 for Apple II compatability, as well as TRS-80s, a Sanyo 555, a Radio Shack Model 100 portable, etc. So, if you would like, you may send in any interesting programs in support of problems or solutions on disk and most of the time I will be able to accommodate you.

One other new development in the technological field: calculators doing symbolic math. In early March 1987, as a member of AMATYC's Academic Computing Subcommittee, I received a note about calculators with such capability, but no information. About the same time, the Mathematical Association of America (MAA) came out with a story in its *Focus* newsletter concerning the HP-28C, by Hewlett Packard.

Needless to say, I did my part as a good American consumer and rushed out to buy one. (What do you expect from a guy with 9 computers, 3 printers, and 2 photocopy machines in his home??)

It is exactly as described in the *Focus* piece. It does take about 10 hours to learn to use reasonably well, as stated in the article. It is powerful. It also is, however, of somewhat limited practical use for a mathematician, but I have managed to use it in my Linear Algebra classes as a little tool for solving matrix equations AX = B. I'm not sure that my $180 (discount price) was well spent, but maybe I'm jaded after seeing the thousands of dollars of review software I get each year.

Enough about computers. Let me just reiterate an offer from a previous issue, if you are interested in the pleasurable aspects of computing (given that this column is not really the most appropriate place). Send me $1 to defray expenses of postage and photocopying, and I'll send you a sample issue of REC, *The Recreational & Educational Computing Newsletter*. Just send it to me at the same address as used for problems and solutions, below.

Okay, on with the customary business of a problem section. I remind you that the **AMATYC Review** Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics, but particularly those which have applicability of some kind in the lives of two-year college math faculty and students. Please eschew routine and familiar problems, however. If you're not sure, send something in anyway. At the least, we may be able to modify it to turn a routine question into a real think piece.

Send your problems, solutions, comments, generalizations, special cases, observations, improvements, and bank books to:

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
129 Carol Drive
Clarks Summit, PA 18411

Just to make sure, I'm still using my home address, but it is highly likely that I will remain with Penn State (permanently).

And no, silly! Of course you don't really have to send your bank books. But, we can all fantasize just a bit, can't we? (Besides, that was an alertness test.)

Let's bring on the problems . . .

Set M Problems. Due for ordinary consideration March 1988. Of course, no problem is ever closed permanently, and new insights are always welcome. Plus, if there is a production delay, we may extend the deadline appropriately.

M-1. Proposed by Steven Blasberg, West Valley College, Saratoga, CA.

Solve the following doubly-true cryptarithm, assuming (as is customary) that each letter represents a unique decimal digit:

```
    SIX
+ SEVEN
+ SEVEN
-------
TWENTY
```

M-2. Proposed by Steven Blasberg.

a) Let (a,b,c) be a Pythagorean triple with $a^2 + b^2 = c^2$. If (a,b,c) $\begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ = (d,e,f), prove that (d,e,f) is another such triple.

b) Moreover, show that if (a,b,c) is primitive, then (d,e,f) is, too. (I.e., it inherits the property somehow. Note that this is usual matrix multiplication.)

Problem Editor's Comment: There have been papers on this, but this is surely not that well known, and the problem looks good. But if that were not enough, take a look at the next problem.

M-3. Proposed by Stephen Plett, Fullterton College, Fullerton, CA.

We all know that a Pythagorean triple (a,b,c) of positive integers with b = a — thus inducing an angle of 45 degrees — is impossible. However, given any challenge tolerance, prove that there exists a Pythagorean triple representing a triangle, one of whose angles comes arbitrarily close to 45 degrees in measure.

M-4. Stolen by me from my colleague, John Graham, Pennsylvania State University, Wilkes-Barre Campus, Lehman, PA.

Evaluate the series: $\sum_{k=1}^{\infty} \frac{k(k+1)}{3^k}$.

Problem Editor's Comment: A "fast and dirty" Basic program will give the answer in seconds. But remember, we want a solution, not just an answer, and we do want mathematics, not just programming. However, a program will serve to verify your answer by other means.

Problem M-5. Proposed by the Problem Editor.

Consider the equations: "1/2 + 1/50 = .52" (correct), and "1/3 + 1/7 = .21" (nonsense). When is it true, for a and b integers, that 1/a and 1/b = .(a+b)? (Here, the right member means the number constructed from a decimal point followed by the integer which is the sum of a and b.)

Problem M-6. Proposed by Stephen Plett.

Using only a compass — no straightedge or other tools — show how to determine the vertices of a square.

Solutions

No solutions were received for I-3. Perhaps it was too hard?

Easier Collatz-Like Problem

J-1. Proposed by the Problem Editor. (paraphrased) Show that sufficient repetition of the function g:Z --->Z will always produce the number 1, where g(n) is defined to be 2n + 2 iff n is odd, and n/2 iff n is even. (As usual, Z = {integers}.)

Solution by Lawrence Sher, Manhattan Community College, NY, NY, with similar solution by the Problem Editor.

First, consider an odd number, 2p+ 1. Then g(2p+ 1) = 2(2p+ 1)+ 2 = 4(p+ 1), which is even. This will map into p+ 1 after two further iterations of g. Note that p+ 1, whether odd or even, is less than 2p + 1. Thus, any odd number will always eventually map into a smaller odd number $\leq$ p + 1.

By infinite descent, any odd number will eventually map into 1.

Next, note that any even number will eventually map into an odd number smaller than the even number.

Hence, all numbers will, upon sufficient repetition of g, eventually map into 1.

I will present a reader's solution to J-5 next issue. I also mentioned that K-2 was covered in MACE, but I'll try to have something next issue.

Linear Compositions Need Not Come From Linear Functions

K-3. Proposed by the Problem Editor, Penn State University, Wilkes-Barre.

If f(f(x)) is linear, must f itself be linear? Are there simple hypotheses that will ensure f being linear, such as continuity, monotonicity, etc.?

Solution by John Graham, Penn State University, Wilkes-Barre.

The answer is NO. For a counterexample, let $f(x) = \sqrt{1 - x^2}$ for $x \in [0, 1]$. Then $f(f(x)) = x$ (check this, noting that $0 \leq x \leq 1$), so f of f is linear, monotone, and differentiable, but f — whose graph is the quarter-unit-circle in the first quadrant — is continuous, monotone, and differentiable, but not linear.

Problem Editor's Comment: Note that John's solution uses a self-invertible function f to get the linearity (i.e., we have $f(f(x)) = x$). Clever!

That is all for now. Enjoy, and remember: may all your problems be purely mathematical!

We have the best books for your students—

AVAILABLE NOW

ALGEBRA AND TRIGONOMETRY, Second Edition, by Munem and Foulis
COLLEGE ALGEBRA, Second Edition, by Munem and Foulis
COLLEGE TRIGONOMETRY, by Munem and Foulis
PRECALCULUS Functions and Graphs, Fourth Edition, by Munem and Yizze
CALCULUS, Second Edition, by Munem and Foulis

IN JANUARY 1988

BEGINNING ALGEBRA, Fourth Edition, by Munem and Tschirhart
INTERMEDIATE ALGEBRA, Fourth Edition, by Munem and Tschirhart
ALGEBRA FOR COLLEGE STUDENTS, Second Edition, by Munem and Tschirhart

WORTH PUBLISHERS, INC. 33 Irving Place, NY, NY 10003

SOFTWARE REVIEWS

Edited by Shao Mah
Red Deer College
Red Deer, AB, Canada T4N5H5

TITLE: **MAGIC MATH PLUS**
AUTHOR: DR. MICHAEL W. ECKER
PUBLISHING CO.: RECREATIONAL MATHEMAGICAL SOFTWARE
129 CAROL DRIVE
CLARKS SUMMIT, PA 18411 (717) 586-2786

COST: $27.50 U.S. ($77.50 LICENSED)

SOFTWARE: 1 disk, single-sided, double density

The MSDOS version was evaluated on an IBM. Two more versions exist, one for TRSDOS and one for SANYO 550 or 555.

HARDWARE REQUIREMENTS:

Depending on the version, this can be run on any MSDOS computer with at least 64K and one disk drive or TRS-80 Model 3, 4, 4P or 4D with at least 32K and one disk drive.

DOCUMENTATION:

This consists of a 4-page description of the different versions, how to back up the disk, and how to load. The program is menu-driven and instructions are included within each program. Thus, instructions are not required in the documentation. However, it would be useful to include the MAIN MENU in the documentation (see CONTENT).

CONTENT: The disk contains 17 programs. A copy of the main menu is attached at the end of this report. The programs can be divided as follows:

- 4 number tricks (#1, 2, 12, 15)

  The user chooses a number and, according to which program is being run, either the user or the computer manipulates the number. Then (like magic), the computer displays the number chosen. Where an explanation exists, it is clear and explains the concept first in prose, then algebraically.

- 5 games (#3, 4, 8, 13, 14)

  All games are played user vs. computer. #14 may be played with 2 people.

  The games vary in interest level. For example, #3, a higher-order of blackjack, or 21, has too much instruction for very little game-

playing. In addition, user involvement is limited since score-keeping is instantaneous. The display can be made more attractive. Game #4 is fun. This is a higher-order version of tic-tac-toe. The game board is a 4 x 4 grid with 4 positions within each space. The object is to get 4 sets of "4 in a row." This is useful in helping students think in a multi-dimensional mode. There is an option to escape, if the play becomes too long. Games #8 and #13 are variations of the counterfeit gold coin puzzle. The explanation is too much for the amount of game-playing. For these games, it is too easy to "play" by hitting any key and not thinking. The user can get stuck in game #13 and have to re-boot the disk.

In game #14, a goal of N, an integer, is set. Each player in turn chooses a number to add to the running total. The one who comes closest without going over wins. While the program indicates that there is a strategy for winning, it provides no hints or explanation.

- 7 table-generating programs (#5, 6, 9, 10, 11, 16, 17)

  These are self-explanatory from the titles. The user inputs initial values according to parameters described within each program. The program then generates the appropriate table. #16 demonstrates a self-generating program. #17: The conjecture: Any number can be made to approach 1 if the following is successfully applied: Odd — multiply by 3 and add 1. Even — divide by 2. This program displays all the iterations required from a chosen number to 1.

- 1 simulation (#7)

  The program goes through a simulation of door-opening and closing (cf. binary digits). "There are 25 prison cells. There are 25 moments of time. At the start, all cells are initially closed. Then, at moment #1, all cells are opened. At moment #2, only those cells whose numbers are multiples of 2 are reversed, in this case to closed. At each moment K, each cell with a number which is a multiple of K is reversed. Which cells are open after 25 moments?"

  The actual simulation happens too quickly to notice a pattern. The explanation is good but probably not of much interest to a remedial group.

GENERAL EVALUATION:

Dr. Ecker displays his talent at recreational computing through this potpourri. The games represent some of the classic problems in mathematics; the table-generating programs demonstrate the practicality of the computer.

Use of this in a school setting is very limited. Some of the explanations are too complex for remedial students (eg., #7). The content does not relate directly to course material. Since applicability is non-existent, this would be used during students' spare time, either in class or outside of class, as the instructor saw fit.

Technically, most of the programs were bug-free. In #12 and #13, the program had to be re-booted, since the user could not return to the main menu.

SUGGESTIONS FOR IMPROVEMENT:
Documentation to include the main menu.
Displays could be made more attractive. For example, in game #3, have the "cards" look like cards rather than a list of numbers.

Reviewed by Rosalind Wild, Lakeland College, Vermilion, Alberta, Canada

MAGIC MATH MENU

1) THE BASE TWO CARD TRICK (with explanation)
2) DIGIT PREDICTION TRICK (with explanation)
3) SUPER-BLACKJACK, THE GAME OF 110
4) THE GAME OF SCORE 4
5) FASTLOAN - LOAN AMORTIZATION PROGRAM
6) COMPOUND INTEREST PROGRAM
7) THE ECCENTRIC JAILER — SIMULATION PLUS EXPLANATION
8) THE FAMOUS 12 COINS PUZZLE — A COMPUTER SOLUTION
9) SUPER FAST PRIME NUMBER GENERATOR
10) DAY FROM DATE PROGRAM
11) ADDITIVE SEQUENCES (such as Fibonacci's)
12) SUPER - TRICK! (with explanation)
13) THE GOLD COINS — PUZZLE
14) THE GAME OF N
15) ANOTHER TRICK
16) THE MYSTIQUE OF SELF-REFERENCE
17) THE COLLATZ CONJECTURE

GRAPHING AND FINDING ZEROS

AUTHOR:	THOMAS SHILGALIS
ADDRESS:	ILLINOIS STATE UNIVERSITY, NORMAL, IL 61761 U.S.A.
PRICE:	FREE (WITH SUBMISSION OF A BLANK DISKETTE)

The software "Graphing and Finding Zeros" by Thomas Shilgalis for Apple II computers contains many excellent mathematics programs for users.

For example, the program "SKETCH" permits the user to graph one, two or three functions on the same set of axes with a choice of either rectangular or polar equations. This capability is very useful for demonstrating the difference of two or three functions. Other graphic programs include parametric equations (PARAMETRIC), integration (INTEGRATE) and double integrals (MULTINTEGRATE). The program, ZEROS, is another graphic program on finding roots for any given function and provides a good teaching aid in equation solving.

The diskette also contains other programs such as derivatives, limits, matrix inversion, solution of linear equations, eigenvalues, related rates, sequences, series, power series, arc length, probability and statistics. Nearly all the programs are interactive and suitable for classroom demonstration or individual use.

The flexible choice of different programs and some of the special features on this software are not usually provided in many existent commercial software. Readers who are interested in obtaining a copy of this software are encouraged to send a blank diskette to Professor Shilgalis.

***(Editor's Note:** This software description was submitted by Thomas Shilgalis in response to a request by the editor. It is intended for informational purposes.)*

Quality you can count on

88

Lange, Rousos, Mason
Mathematics for Business and Consumer, *4th Edition*

Plane, Oppermann
Business and Economics Statistics, *3rd Edition*

Van Matre, Gilbreath
Statistics for Business and Economics, *3rd Edition*

BPI Business Publications, Inc.
1700 Alma, Suite 390, Plano, TX 75075-6916 (214) 422-4389

INSTITUTIONAL MEMBERSHIP

Colleges, publishers, and other institutions can support AMATYC by being Institutional Members. As indicated below, the annual membership fee is $75. Subscriptions to **The AMATYC Review** and **The AMATYC News** are included in this fee. Of particular importance to collegiate institutional members will be the new AMATYC committee for mathematics department chairpersons.

An additional benefit of an institutional membership is one complimentary AMATYC convention registration. Future conventions, which are held in the Fall, will be in Kansas City (1987) and Calgary (1988). Institutional members support the excellence of the programs at these annual conventions.

INSTITUTIONAL MEMBERSHIP APPLICATION

The American Mathematical Association of Two-Year Colleges

Mail to:
Karen Sharp, AMATYC Treasurer, Mott Community College, Flint, Michigan 48502

AMATYC COLLEGE CONTACT _____ POSITION _____

NAME OF INSTITUTION _____

ADDRESS _____

CITY _____ STATE _____ ZIP _____

Membership Fee - $75 (includes **The AMATYC Review, The AMATYC News,** and one complimentary convention registration)

Check One:

☐ Name and title of college official to whom invoice should be sent:

NAME _____ TITLE _____

☐ Payment of $75 is attached.

Please make check payable in US funds to AMATYC.

WHY JOIN AMATYC?

The American Mathematical Association of Two-Year Colleges (AMATYC) was established in 1974 to provide a unique, national forum for two-year college mathematics educators. Today, AMATYC is the only national organization that exclusively serves the needs and purposes of this group.

AMATYC holds a national convention annually in a major city. AMATYC encourages two-year college mathematicians to assume responsible leadership positions; to interact on an equal basis with four-year university personnel; to be members on a proportional basis of national steering or policy committees; to represent at the national level the concerns of two-year college mathematics educators. *The AMATYC Review* provides an opportunity to publish articles by and for two-year college faculty. A national newsletter is published three times yearly for all members. AMATYC is a member of the National Conference Board of the Mathematical Sciences.

CUT HERE. MAIL TO TREASURER

INDIVIDUAL MEMBERSHIP APPLICATION The American Mathematical Association of Two-Year Colleges

MAIL TO: Karen Sharp, AMATYC Treasurer, Mott Community College, Flint, MI 48503

MEMBER'S NAME _____ POSITION _____

COLLEGE _____

COLLEGE ADDRESS _____ PHONE _____

CITY _____ STATE _____ ZIP _____

RESIDENCE ADDRESS _____ PHONE _____

Indicate preferred mailing address: ☐ College ☐ Residence

All payments in U.S. funds. Membership fee:

- ☐ $25 Yearly Individual Membership (any person interested in two-year college mathematics)
- ☐ $10 Yearly Associate Membership (full-time student)
  Name of AMATYC Sponsor _____
- ☐ $22 Reduced Yearly Membership for members of organizations affiliated with AMATYC (see below)
- ☐ $500 Life Membership

Please check appropriate state affiliation:

☐ ArizMATYC (Arizona)	☐ KYMATYC (Kentucky)	☐ OJCMA (Oklahoma)
☐ ARKMATYC (Arkansas)	☐ MMATYC (Maryland)	☐ PSMATYC (Pennsylvania)
☐ CMC¹ (California)	☐ MICHMATYC (Michigan)	☐ SOCAMATYC (South Carolina)
☐ CMC¹ South (California)	☐ MOMATYC (Missouri)	☐ TMATYC (Tennessee)
☐ MATYCONN (Connecticut)	☐ NEMATYC (New England)	☐ TexMATYC (Texas)
☐ FTYCMA (Florida)	☐ MATYCNJ (New Jersey)	☐ VMATYC (Virginia)
☐ IMACC (Illinois)	☐ OHIOMATYC (Ohio)	☐ WAMATYC (Washington)
☐ IRMC (Indiana)		☐ WYMATYC (Wyoming)

Membership includes *The AMATYC Review* and the *AMATYC News*. In addition, the following journals are available at extra cost:

- ☐ Mathematics and Computer Education Journal $16 for 3 issues
- ☐ The College Mathematics Journal $17 for 5 issues
- ☐ UMPA Journal $25 for 4 issues (U.S. subscriptions only)

Total Amount Enclosed _____ Date Paid _____

Make check or money order payable in U.S. funds to AMATYC

EDITORIAL PANEL

Agnes Azzolino	Middlesex County College	Edison, NJ
Ione Boodt	Univ. of Indianapolis	Indianapolis, IN
Mike Davidson	Modesto J.C.	Modesto, CA
Joyce S. Friske	Oklahoma State Univ.	Stillwater, OK
Brian Hickey	East Central College	Union, MO
Gordon W. Hoagland	Ricks College	Rexburg, ID
Nelson G. Rich	Nazareth College	Rochester, NY
Martha M. Wood	Clayton State College	Morrow, GA

REVIEWER PANEL

Darrell H. Abney	Maysville, C.C.	Maysville, KY
Dennis Adams	State Technical Institute	Knoxville, TN
Paul E. Bland	Eastern Kentucky University	Richmond, KY
Wilfred E. Boykin	Univ. Central Ct. State	New Britain, CT
Randall Brian	Vincennes University	Vincennes, IN
Pansy Brunson	Western Kentucky University	Bowling Green, KY
Alan Chutsky	Queensborough C.C.	Bayside, NY
John DeCoursey	Vincennes University	Vincennes, IN
Al Derin	Truman College	Chicago, IL
George Dorner	Harper College	Palatine, IL
David Dyer	Prince Georges	Largo, MD
Joseph R. Fiedler	Ohio State University	Columbus, OH
Gregory D. Foley	Ohio State University	Columbus, OH
Barbara R. Gab	Prince Georges	Largo, MD
Joan Golliday	Santa Fe C.C.	Gainsville, FL
Anne Grams	Embry Riddle Aeronautical Univ.	Daytona Beach, FL
Robert J. Greger	Harford C.C.	Bel Air, MD
Ronald J. Harkins	Miami University	Hamilton, OH
Peter Herron	Suffolk C.C.C.	Selden, NY
Berward G. Hoerbelt	Genesee C.C.	Batonia, NY
Calvin J. Hoh	Paul D. Camp C.C.	Franklin, VA
James T. Johnson	Modesto Jr. College	Modesto, CA
Deborah Levine	Nassau C.C.	Garden City, NJ
Ved P. Madan	Red Deer College	Alberta, CANADA
Dominic Magno	Harper College	Palatine, IL
Pamela E. Matthews	Willamette University	Salem, OR
Mary McCarty	Sullivan C.C.C.	Loch Sheldrake, NY
Beverly Mugrage	University of Akron	Akron, OH
Terry A. Nyman	UW-Fox Valley	Menasha, WI
Carol Olmstead	Univ. of Nevada	Reno, NV
Loken W. Pixley	Richland C.C.	Decatur, IL
Dennis Reissig	Suffolk C.C.C.	Selden, NY
Jack W. Rotman	Lansing C.C.	Lansing, MI
Daren J. Schroeder	Bentley College	Waltham, MA
Joseph A. Smith	Olive Harvey College	Chicago, IL
Lowell Stultz	Kalamazoo Valley C.C.	Kalamazoo, MI
Leslie R. Tanner	The College of Idaho	Caldwell, ID
Travis Thompson	Harding University	Searcy, AR
August J. Zarcone	College of DuPage	Glen Ellyn, IL

AMATYC Institutional Members

Addison-Wesley Publishing College	Reading, MA	01867
Anchorage Community College	Anchorage, AK	99504
Austin Community College	Austin, TX	78768
Bessemer State Technical College	Bessemer, AL	35020
Brooks-Cole Publishing Co.	Monterey, CA	93940
Burlington County College	Pemberton, NJ	08068
Butte College	Oroville, CA	95965
C. S. Mott Community College	Flint, MI	48503
Casper College	Casper, WY	82601
Comm. Coll. Allegheny County	Pittsburg, PA	15237
Chabot College	Hayward, CA	94545
Charles Co. Community College	La Plata, MD	20646
Chemeketa Community College	Salem, OR	97309
Clackamas Community College	Oregon City, OR	97045
Clark Technical College	Springfield, OH	45501
Columbus Technical Institute	Columbus, OH	43215
D.C. Heath and Co.	Lexington, MA	02173
De Anza College	Cupertino, CA	95014
DeKalb Community College-South	Decatur, GA	30034
Delaware Tech. & Comm. College	Newark, DE	19702
EN Boekhandel B V	1000 GC Amster, Hol, Netherland	
Florida College	Temple Terra, FL	33617
Florida Jr. College-Jacksonville	Jacksonville, FL	32205
Fullerton College	Fullerton, CA	92634
George Brown College	Toronto, Ontario, Canada	M5T 2T9
H&H Publishing Company, Inc.	Clearwater, FL	33575
Harford Community College	Bel Air, MD	21014
Harper College	Palatine, IL	60067
Hartnell College	Salinas, CA	93901
Houghton Mifflin Company	Boston, MA	02107
Hudson County Community College	Jersey City, NJ	07306
Idaho State University	Pocatello, ID	83201
Illinois State University	Normal, IL	61761
Johnson Co. Community College	Overland Park, KS	66201
Lakeland College	Vermilion, Alberta, Canada	TOB 4MO
Lane Comm. College/Dept. of Math	Eugene, OR	97405
Lorain Community College	Elyria, OH	44035
Matanuska-Susitna Community College	Palmer, AK	99645
Monroe Community College	Rochester, NY	14623
Moraine Valley Community College	Palos Hills, IL	60645
New Mexico Military Inst.	Roswell, NM	88201
North Idaho College	Coeur D'Alene, ID	83814

North Orange County Comm. College	Fullerton, CA	92634
Northampton Area Community College	Bethlehem, PA	18017
Northern Arizona University	Flagstaff, AZ	86011
Penn State Univ.-Capitol Campus	Middleton, PA	17057
Pennsylvania State University	Dunmore, PA	18512
Pikes Peak Community College	Colorado Springs, CO	80906
Polk Community College	Winter Haven, FL	33881
Portland Community College	Portland, OR	97219
Prestonburg Community College	Prestonburg, KY	41653
Raymond Walters College	Cincinnati, OH	45236
Richland Community College	Decatur, IL	62526
Ricks College	Rexburg, ID	83440
Robert Morris College	Coraopolis, PA	15108
Saunders College Publishing Co.	Philadelphia, PA	19105
Schoolcraft College	Livonia, MI	48152
Scott Foresman Company	Glenview, IL	60025
Somerset County College	Somerville, NJ	08875
Southwest Texas State University	San Marcos, TX	78666
Southwest Virginia Comm. College	Richlands, VA	24641
St. Bernard Parish Comm. College	Chalmette, LA	70043
Stark Technical College	N. Canton, OH	44720
SUNY College of Technology	Alfred, NY	14802
Tarrant Community College	Hurst, TX	76054
Three Rivers Community College	Poplar Bluff, MO	63901
University of Cincinnati	Cincinnati, OH	45221
University of Puerto Rico	Carolina, PR	00630
University of Puerto Rico	Bayamon, PR	00620
Volunteer State Community College	Gallatin, TN	37066
Wadhams Hall College	Ogdensburg, NY	13669
Wallace State Community College	Selma, AL	36702
Waterbury State Technical College	Waterbury, CT	06708
West Publishing Company	Moss Beach, CA	94038
Westark Community College	Ft. Smith, AR	72913

American Mathematical Association of Two-Year Colleges

EXECUTIVE BOARD OFFICERS
1985-1987

President
Stephen B. Rodi
Austin Community College
P.O. Box 2285
Austin, TX 78768
(512) 495-7222

President-Elect
Karl J. Smith
Santa Rosa Junior College
Santa Rosa, CA 95401
(707) 527-4356

Treasurer
Karen Sharp
Mott Community College
Flint, MI 48502
(313) 762-0279/0277

Secretary
Cheryl Cleaves
State Technical Institute
Memphis, TN 38134
(901) 377-4110

Past President
Amber Steinmetz
Santa Rosa Jr. College
Santa Rosa, CA 95401
(707) 527-4706

REGIONAL VICE PRESIDENTS

Northeast
Allen R. Angel
Monroe Community College
Rochester, NY 14623
(716) 424-5400 Ext. 5215

Mid-Atlantic
Donna M. Szott
Allegheny County C.C.-So.
West Mifflin, PA 15122
(412) 469-1100 Ext. 360

Southwest
Joyce S. Friske
Oklahoma State Univ.
301 Gundersen
Stillwater, OK 74078
(405) 624-7125

Southeast
Margie Hobbs
State Technical Institute
Memphis, TN 38134
(901) 377-4110

Central
Ron Beeler
East Central Jr. College
P.O. Box 529
Union, MO 63084
(314) 583-5193

Northwest
Steven S. Terry
Ricks College
Rexburg, ID 83440
(208) 356-1406

Midwest
Dale Ewen
Parkland Community College
2400 West Bradley Ave.
Champaign, IL 61821
(217) 351-2225

West
Herbert E. Garrett
Long Beach City College
Long Beach, CA 90808
(213) 420-4476/4424

ACADEMIC COMMITTEES

Developmental Mathematics
Charles Luttrell
Frederick Comm. College
Possumtown Pike
Frederick, MD 21701

Student Mathematics League
Terry Shell
Santa Rosa Jr. College
1401 Mendocino Ave.
Santa Rosa, CA 95401

Technical Mathematics
Ray E. Collings
Tri-County Tech. College
P.O. Box 587
Pendleton, SC 29670

Academic Computer
Kathleen Thestia-Preece
Phoenix College
1202 W. Thomas Road
Phoenix, AZ 85013

Encouragement of Women
Marilyn Mays
North Lake College
5001 MacArthur Blvd.
Irving, TX 75308

Education
Geoffrey Akst
Bor. of Manhattan C.C.
199 Chambers St.
New York, NY 10007

ERIC Clearinghouse for Junior Colleges OCT 14 1988

THE AMATYC REVIEW

Published by the
AMERICAN MATHEMATICAL ASSOCIATION OF TWO-YEAR COLLEGES

VOLUME 9, NUMBER 2 SPRING 1988

FEATURING:

Encouraging Clowns — Two-year college teachers Kylene Norman and John Bailey "clown it up" to motivate middle school students to study mathematics. They are the authors of

"The Clark TAC Computer: A Sweet Way to Encourage Kids to Stay in Mathematics"

ALSO IN THIS ISSUE:

— An Induction Proof of the Inequality of the Weighted Geometric and Arithmetic Means
— Pre- and Post-Instructional Strategies for the College Developmental Mathematics Class
— Numerical Integration: Surprises from the Trapezoidal Rule

BEST COPY AVAILABLE

The Official Journal of the

American Mathematical Association of Two-Year Colleges

PURPOSE

AMATYC remains committed to its original purposes:

- To provide a national forum for the exchange of ideas
- To further develop and improve the mathematics education of students of two-year colleges
- To coordinate activities of affiliated organizations on a national level
- To promote the professional development and welfare of its members

The AMATYC Review provides an avenue of communications for all mathematics educators concerned with the views, ideas and experiences pertinent to two-year college teachers and students.

SUBMISSION OF MANUSCRIPTS

MANUSCRIPTS MUST CONFORM TO THE FOLLOWING FORMAT:

They must be typewritten on one side and double-spaced on 8½" x 11" paper. Five copies of each manuscript, which has not been published before, should be submitted to the editors. If the manuscript contains any photographs, they should be black and white glossy prints. The first page of the manuscript should contain the article's title, all the authors' names and their institutional affiliation along with the name and address of the person to whom all correspondence should be mailed. Authors are advised to consult the *Publication Manual of the American Psychological Association* (3rd ed.). A guideline for authors is available from the editor.

Manuscripts should be sent to: Don Cohen, Editor, *The AMATYC Review*, SUNY College of Agriculture and Technology, Cobleskill, NY 12043. Any manuscript not accepted for publication will be returned.

ADVERTISING

For information concerning advertising rates and schedules, write to Eleanor Young, Sinclair Community College, Dayton, OH 45402 or telephone (513) 226-2789.

STAFF

Editor:	Don Cohen, SUNY Ag-Tech College, Cobleskill, NY 12043, (518) 234-5116
Production:	Roy Cameron, SUNY Ag-Tech College, Cobleskill, NY 12043, (518) 234-5351
Advertising:	Eleanor Young, Sinclair Community College, Dayton, OH 45402
Artwork:	Robert Burghardt, Rockland Community College, Suffern, NY 10901

PUBLICATION

The AMATYC Review is published twice a year in September and February.

ISSN 0740-8404

A proud publishing tradition continues.

Addison-Wesley Mathematics, 1988:

Calculus and Analytic Geometry, Seventh Edition (16320), by George B. Thomas, Jr. and Ross L. Finney

Calculus: A Modeling Approach, Fourth Edition (12216), by Marvin L. Bittinger

Applied Calculus, Second Edition (12211), by Marvin L. Bittinger and Bernard B. Morrel

Essential Mathematics, Fifth Edition (14290), by Mervin L. Keedy, Marvin L. Bittinger, and William Rudolph

Developmental Mathematics, Second Edition (14354), by Mervin L. Keedy and Marvin L. Bittinger

Discrete Mathematics (10628), by Jim Bradley

Linear Algebra (13089), by Carroll Wilde

Technical Mathematics, Volume I (15408), & **Volume II** (15409), by Thomas A. McHale and Paul T. Witzke

Differential Equations: An Introduction, Second Edition (15407), by David A. Sanchez, Richard C. Allen, Jr., and Walter T. Kyner

Elementary Number Theory, Second Edition (11958), by Kenneth H. Rosen

Introduction to Probability and Statistical Applications, Third Edition (15465), by Paul Meyer and Stephen Fienberg

Introductory Mathematical Statistics (19310), by Peter Zehna

Looking for advanced mathematics and physics titles? Call or write for a free catalog of books in our *Advanced Book Program.*

Contact your Addison-Wesley sales representative for more details on any of the titles you see in this ad.

Addison-Wesley Publishing Company
Reading, MA 018[illegible] • (617) 944-3700 • *The leader in mathematics.*

The AMATYC REVIEW Spring, 1988 Volume 9, Number 2

Editorial ..... p. 5

VIEWPOINT

The Clark TAC Computer: A Sweet Way to Encourage Kids to Stay in Mathematics ..... p. 7
by John W. Bailey and Kylene E. Norman

MATHEMATICAL EXPOSITION

Using the Number Line to Factor Quadratics ..... p. 12
by D. Irene Fell Rodgers

Numerical Integration: Surprises from the Trapezoidal Rule ..... p. 18
by Travis Thompson

Bilateral Symmetry of the Hypergeometric Distributions ..... p. 21
by Sandra M. Pulver

A Recursive Algorithm for Solving Linear Diophantine Equations ..... p. 26
by Gene P. Blandford

An Induction Proof of the Inequality of the Weighted Geometric and Arithmetic Means ..... p. 31
by Terence Brenner

SHORT COMMUNICATIONS

The Derivative of $\sin(kx)$ — Why the Chain Rule? ..... p. 38
by Ann D. Holley

ACROMATYCS ..... p. 42
by Joseph Browne

MATHEMATICS EDUCATION

Pre- and Post-Instructional Strategies for the College Developmental Mathematics Class ..... p. 46
by Joannis Kelly Flately, Dennis Gittinger and Adele Ducharme

Reflections on the Studies of Comparisons of US Students and Students in Other Countries ..... p. 51
by James Baldwin

REGULAR FEATURES

Educational Reflections ..... p. 55
by Gregory D. Foley
Pandora's Rectangular Parallelipiped ..... p. 62
by Leslie R. Tanner
Book Reviews ..... p. 66
by John J. Edgell
Problem Section ..... p. 70
by Michael W. Ecker
Computer Software Section ..... p. 76
by Shoa Mah
Mathematical Trivia ..... p. 76
by Ralph Selig

Houghton Mifflin 1988

New for 1988

Business Mathematics
Richard N. Aufmann and
Vernon C. Barker
Both of Palomar College
Joanne Lockwood

About 500 perforated pages paperback • Student Computer Disk Computer Tutor™ • Instructor's Annotated Edition • Instructor's Manual • MicroTes': A Computerized Test Bank (IBM®) • Call-in Test Service • GPA: Grade Performance Analyzer • Just published

In a convenient worktext format, *Business Mathematics* covers not only basic mathematics and the use of business machines, but also essential study skills. The authors treat both hand-held calculators—with a section in each chapter—and desk-top machines.

Contemporary in approach, the worktext reflects the impact of the computer and includes an overview of software programs for business.

Now Available

Practical Math for Business
Fourth Edition
Alan R. Curtis

382 three-hole punched, perforated pages • paperback • Instructor's Annotated Edition • Transparencies • 1987

Curtis's Fourth Edition features revised examples and problems and adds new transparencies to the extensive support package.

For adoption consideration, request examination packages from your regional Houghton Mifflin office.

Houghton Mifflin

13400 Midway Rd., Dallas, TX 75244-5165
1900 S. Batavia Ave., Geneva, IL 60134
925 E. Meadow Dr., Palo Alto, CA 94303
University Sq., 101 Campus Dr.,
Princeton, NJ 08540

Editorial:

by Don Cohen

The Process of Manuscript Development

You worked very hard on that article that you just submitted for publication in *The AMATYC Review*. What can you expect in return from the editor? Common replies might be, an acknowledgement that the article has been received and sent out for review, and a reasonably fast decision on whether or not the article is acceptable for publication. Well, I hope that I was not selected to be the editor of this journal because people thought that I was good at shuffling papers or tallying the opinions of reviewers to reach an "accept" or "reject" decision; these are important editorial tasks, but they do not require the expertise of a mathematics educator.

During my first year as editor, I received only a few articles that the reviewers and I thought were either completely acceptable for publication as submitted, or were acceptable with some minor technical changes or revisions from the author. On the other hand, I also received only a few articles that I thought were inappropriate for publication. Usually, the initial submission of an article is the first step in the *process* of article development. I see my main editorial function as organizing my opinion of an article with the opinions of three reviewers into a developmental review that is designed to help the author revise his manuscript so that it is a more valuable reading experience for two-year college educators.

In addition to comments and questions on the mathematics or language usage, developmental reviews usually contain suggestions from among the following areas:

1. Articles should be worded concisely and contain no material that does not add value to the topic.
2. Articles should be complete. It might be appropriate to include additional materials to enrich the topic under consideration.
3. Articles should be interesting and readily understandable to the sophisticated reader. Developmental reviews frequently contain suggestions for "perking-up" articles and for clarifying what the editor considers to be confusing statements.

Frequently, I think of myself as the proverbial "white haired, scholarly college professor" who is working with a group of students in the development of their term papers. However, these thoughts are wrong in several respects: first, I do not have white hair; but more importantly you are not my students and you are certainly more scholarly about your topic than I am. Realistically, the developmental review is meant to offer you suggestions (not mandates) for enhancing a valuable manuscript. Authors should discuss these suggestions with their colleagues or give me a call.

Finally, it is hoped that the author of a returned manuscript will submit a revised version of the manuscript that can again be carefully considered for publication. Your effort and thoughts are appreciated.

Attracting Bright Mathematics Student to Two-year Colleges

In the past few years the number of lower level mathematics sections has increased and the number of calculus sections has decreased at the State University of New York College of Agriculture and Technology at Cobleskill. While working with students at the lower end of the ability continuum is a challenging and rewarding experience, working with students at the other end of the ability continuum is equally challenging and rewarding (and frequently more satisfying to our egos).

During my first year as editor of this journal, I received many articles on how schools are meeting the mathematical needs of their low ability students; however, I did not receive any articles on any special program for attracting bright mathematics students to our schools. If your college is doing anything special in this area, I invite you to put it into manuscript form and submit it for publication in *The AMATYC Review*.

A *Trivial* Comment

It is my pleasure to welcome Ralph Selig, the director of the Computer Learning Center at Fairleigh Dickinson University, to our staff. Ralph will edit a mathematical trivia column that will challenge and entertain our readers. Contributions of trivia items such as not so famous quotes, historical anecdotes and short puzzles may be sent to:

Ralph Selig, Director
Edward Williams College
Computer Learning Center
Fairleigh Dickinson University
Hackensack, NJ 07601

Quality you can count on

88

Lange, Rousos, Mason
Mathematics for Business and Consumer, *4th Edition*

Plane, Oppermann
Business and Economics Statistics, *3rd Edition*

Van Matre, Gilbreath
Statistics for Business and Economics, *3rd Edition*

BPI Business Publications, Inc.
1700 Alma, Suite 390, Plano, TX 75075-6916 (214) 422-4389

THE CLARK TAC COMPUTER: A Sweet Way to Encourage Kids to Stay in Mathematics

by John W. Bailey and Kylene E. Norman
Engineering Technologies Department
Clark Technical College
Springfield, Ohio 45501

John Bailey served 21 years in the Air Force and is in his third year at Clark Technical College. He is a physicist and specializes in electronics and computers. Kylene Norman has been a mathematics teacher for 22 years for all ages from elementary school through college.

Editor's Note: *As mathematics faculty, we can no longer sit back and say that student recruitment is the sole responsibility of our admissions officers. In the following article, John Bailey and Kylene Norman present their views on how faculty can get involved in the recruitment process.*

Nationwide, postsecondary institutions are realizing a need to begin recruiting at an earlier age. Indiana and Minnesota have begun programs to inform middle school students of the advantages of education beyond high school. Mooney (1987) writes, "The Minnesota Higher Education Coordinating Board sends middle schools college-planning booklets to distribute, and this year produced a video tape about the benefits of higher education." She further notes that letters and brochures outlining the benefits of higher education, along with questionnaires, are sent to Indiana ninth graders. Another letter follows to those responding to the questionnaire as well as to those not responding. It is reasonable to believe that informing students about college earlier than in the typical eleventh or twelfth grades might produce more deliberate choices than those now made. Aggressive institutions can actively inform youngsters of their services and programs, if for no other reason than name recognition in later years.

The dwindling number of students pursuing careers in scientific fields is also a concern of today's educators. The study of mathematics seems to be the key here. By the time students leave middle school, they have frequently determined their future in mathematics. They will either head into algebra or end with general mathematics. Many of those entering algebra will opt out of higher mathematics sometime during the high school years.

Clark Technical College's Clark Tac Computer Show is an innovative attempt to keep kids interested in mathematics as well as to serve as a first exposure to the college. The idea for the show was born in the spring of 1986 when the college was planning to host 150 middle and high school students for a day of mathematics and science workshops. The college's engineering technologies department was asked to design one of four concurrent sessions to be offered. Two members of the department, a

mathematician and an engineer, accepted the challenge. Rather than supervise a typical introductory session on keyboard instruction, the pair decided to attempt a low anxiety, "math-can-be-fun" approach to how computers perform mathematics. The result was the Clark Tac Computer Show, and the hour of silliness with a purpose grew in popularity from that first day. Two teachers accompanying the first group to the workshop asked if the show could go on the road. In the 1986-87 school year, it was presented to over 600 area students in groups of approximately 30 each and has been scheduled for 1,500 students for the 1987-88 school year.

The Clark Tac Computer Show

During the show each student is given a computer kit. It consists of two strips of five-cavity heart-shaped plastic candy molds, which act as the registers and answer displays; a box of TicTac candies, which are the bits (or shall we say "bytes?") of the machine; a set of 12 adhesive labels; and a pencil embossed with the college's name. Students are also given a four-page handout which carefully explains the operation, solves examples, and includes a worksheet of problems.

Teachers of classes to which the show is presented must have exposed their groups to only the simplest notion of powers and exponents. Before the show begins, the Clark Tac team asks the teacher about the ability level of the group. The pace of the show is based upon that evaluation. The content is never compromised, but the speed of delivery and language used is adapted to the audience.

An atmosphere emphasizing entertainment is established from the beginning. The team enters the class dressed in hats imprinted with "CTC" (which could mean Clark Technical College or Clark Tac Computer); buttoned-up lab coats, which conceal white tee shirts; black bow ties; white button-down collar shirts; and black slacks. The lab coats are removed during the show to reveal the tee shirts. The mathematician has a "1" on the front and a "0" on the back, while the engineer has just the reverse. Both shirts have "Clark Tac" printed on the left sleeve.

Having faculty appear as clowns sets the mood for the show. The engineer begins with a joke about mathematicians and the mathematician counters with one about engineers. Students pick up from the beginning about the age-old arguments between the two disciplines, yet the students soon realize that both must cooperate to make computers. Students generally wind up taking sides with either the engineer or the mathematician. Jokes and puns are woven throughout the presentation to detract from what could be a fear of failure — they are not given the chance to fail, much less to worry about it.

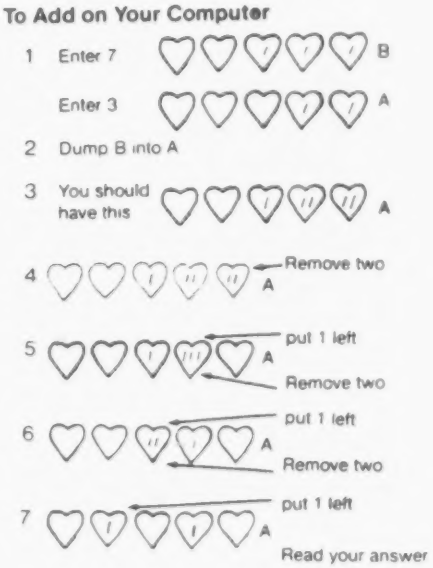
The mathematician begins with a discussion of decimal place holders as powers of ten and then changes the base to two, thereby producing binary place holders. The engineer has the students label five binary place holders with their decimal values on each of the plastic candy molds. The mathematician builds a number in binary using 0's and 1's and the students load the number using a TicTac as a 1 and no TicTac as a zero. Very soon the class has caught onto the idea of adding the place values to determine the value of the number loaded.

At that point the pair remove their lab coats, revealing the zeroes and ones. They either face the students or face away to illustrate four possible binary numbers, while having the class answer a "What are we?" question. Some of the answers are funny, which only adds to the humor of the show.

In the next part of the performance, the mathematician shows the concept of adding binary numbers at the board, complete with the carry process. This is again related to the same notion in base-10 arithmetic. The engineer has the students load the numbers to be added, one number in each register. One register is dumped into the other and students must look at the result. Places that now have no TicTacs or one TicTac are ok, but two or three TicTacs signal that a carry must be made. They are told to remove two TicTacs, put one of them in the place holder to the left (take two out —put one left), and eat the other one. They may also eat mistakes, overflows, and underflows. This addition process mimics what occurs in a real computer, the only difference being the real machines use electric impulses to click on (a one) or off (a zero). Multiplication, division, and subtraction are explained in a similar fashion.

The handout carefully explains each operation. An excerpt of two of the operations follows:*

To Add on Your Computer



To Multiply by 2 on Your Computer

Answer 11010 = 26

To Multiply by 3 on Your Computer

Multiply by 2 (Shift left and dump)
Enter your number once more on top and add

The show closes with the students being referred to the worksheet portion of the handout – a game in which problems in binary can be done on their Clark Tac Comupter. The answers can be decoded by pairing them with corresponding letters of the alphabet to reveal the message, "Compute at Clark Tac." The pair concludes with a final reminder to the students that they have just learned some fairly tricky mathematics, have been successful, should consider the occupational importance of mathematics, and should stay in mathematics. They also issue an invitation to the students to visit the campus and them at any time.

*Interested readers may obtain a complete copy of the Clark Tac handout by sending a request to the authors.

A follow-up "thanks-for-having-us" letter is sent to the teacher immediately following the show. This letter repeats the suggestion to the students to stay in mathematics. Students have gone so far as to write individual thank-you's to the Clark Tac team. But regardless of whether the response is verbal or written, students seem genuinely thrilled with the experience. The team has no way of knowing if the show actually keeps kids interested in mathematics over an extended period of time, but there is little doubt that they find the one hour show a positive experience.

Conclusion

Every year a new batch of students will enter the seventh grade, so there will be an ever-new audience. The show cannot be "played out." Visitors to the show have included the media, school administrators, and state education officials. Papers on the idea have been accepted at statewide and national professional meetings. There has been some interest in marketing the concept nationally. This is a possibility, but colleges need more than a mere understanding of the idea. They also need a team of "clowns" willing to do the performance. The fast pace of the show requires a fun-loving pair, dedicated to seeing that kids laugh and learn at the same time. One works the entire audience as a group, while the other walks the floor, sensitive to any student struggling with a concept. The team must truly enjoy the activity -- forced humor does not work -- ad libs enhance the setting. The show is a physical and emotional drain on the players, but seeing a frustrated child's eyes light up with understanding is worth all the effort it takes.

The college itself has supported the players in class schedule adjustments and financial assistance for the coming year to cover publicity, supplies, travel, and time. Why? Because this activity touches hundreds of potential students and their parents, even though college for these students is four to six years away. The name of the show is a subtle pun on the name of the institution. This name recognition is reinforced on the costumes, pencils, and worksheets. The Clark Tac Computer Show at least gives the students an initial "taste" of college. The rationale here is that this first exposure to Clark Tech has been a pleasant and memorable one.

The rewards of this experience for the schools may well be that students have a renewed interest in mathematics. The reward for the college may be in future recruiting and a positive community image. The teachers and students have a happy break from routine, if nothing else. But the reward for the Clark Tac team has been the room full of 30 smiling faces pleased with themselves for understanding a little about mathematics in computers, determined to decode the message, excited to take their own candy computer home to show Mom and Dad what they can now do.

REFERENCE

Mooney, C. J. (1987). Indiana Tries to Entice More Young People to Consider College. *The Chronicle of Higher Education,* 33 (24) 24.

ACKNOWLEDGEMENTS

The Clark Tac team thanks Clark Technical College and its administration for supporting the Clark Tac Computer Show; Ferraro U.S.A., New York, for providing all the TicTacs for the show; and Dr. Marsha S. Bordner, Chair of General Studies, Clark Technical College, for her time and help in editing this article.

MATHEMATICAL EXPOSITION

Using the Number Line to Factor Quadratics

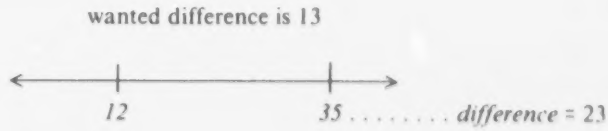
by D. Irene Fell Rodgers
Butte College, Oroville, CA 95965

Irene received her M.S. in Mathematics from Western Washington University in 1968 and has taught at CSU, Chico and Butte College.

Factoring quadratics need not be a time consuming, tedious task. There is a way to factor quadratic expressions that students say is "fun to do." With the help of the number line, it becomes a rather straightforward process that offers very little chance for error. It is even a simple matter to determine whether or not a quadratic expression is prime, and coefficients with multiple factors do not affect the simplicity of the process. For example, students usually have no problem factoring by grouping quadratic expressions such as $12x^2 - 28x + 15x - 35$. The method to be illustrated will take an expression such as $12x^2 - 13x - 35$ and rewrite it as $12x^2 - 28x + 15x - 35$. The problem, obviously, is to find the numbers -28 and 15, and this is a fast simple process with the use of the number line.

If you consider the product $(ax + b)(cx + d) = acx^2 + adx + bcx + bd$, you will note that the factors of the first and last terms are exactly those factors that give you the values for the two middle terms. Therefore, with the above expression $12x^2 - 28x + 15x - 35$, note that the positive factors of 28 and 15 (2, 2, 7, 3, 5) are the same as the positive factors of 12 and 35. It is also necessary to consider the sign of the last term of the quadratic. If it is negative, then the two numbers you seek must have a difference that gives you the coefficient of the middle term. If it is positive, then their sum must give you that coefficient.

So with the expression $12x^2 - 13x - 35$, we start with 12 and 35, put them on the number line and look at their difference:



Since the difference of 12 and 35 is greater than the middle term 13, the desired numbers must be closer together. So the move is made up the number line from 12. Since the only numbers that can be considered are those that can be obtained from the factors of 12 and 35, it is a simple matter to eliminate those numbers that are not valid candidates. Thirteen is thus eliminated. Fourteen is a possibility. Using the factors of 2 and 7 for 14, the product of the remaining factors (3, 5, 2) is 30. The difference of 14 and 30 is 16, still greater than the

desired difference. So the movement up the number line is continued to the next number 15.

wanted difference is 13

12 35 . . . . . difference is 23
14 30 . . . . . difference is 16
15 28 . . . . . difference is 13

Using the factors 3 and 5 for 15, the product of the remaining factors give 28. The difference of 28 and 15 give the desired number 13. Since −13 is desired, −28 and 15 are used to rewrite the expression.

$$\begin{aligned} 12x^2 - 13x - 35 &= 12x^2 - 28x + 15x - 35 \\ &= 4x(3x - 7) + 5(3x - 7) \\ &= (4x + 5)(3x - 7) \end{aligned}$$

It should be noticed that the number line used above is not scaled as would be done were graphing the goal. Here the number line is used to systematically move from number to adjacent number along the line in order to eliminate the possibility of overlooking a possible factor. The *position* of the number (left or right) in relationship to a given number rather than its distance from the given number is the important consideration.

Consider an example where the expression is prime:

$$8x^2 + 10x - 15.$$

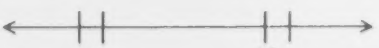
You want to rewrite the coefficient of the middle term 10, as the difference of two numbers. Therefore,

wanted difference is 10

8 15 . . . . . difference is 7

The positive factors of 8 and 15 are 2, 2, 2, 3 and 5. Since the desired difference 10 is greater than that given by 8 and 15, the movement is made down the number line from 8. You move to 7. It cannot be obtained from the given factors. Six can be obtained using 2 and 3, the product of the remaining factors (2, 2, 5) is 20. The difference of these two numbers is 14.

wanted difference is 10



8 15 . . . . . difference is 7
6 20 . . . . . difference is 14

Moving from 8 to 6 on the number line took you from a difference of 7 to a difference of 14. Since 10 is between 7 and 14, it is impossible to obtain a difference of 10 using the given

factors. This indicates that the expression is prime.

Let's condense the process somewhat and factor the quadratic:

$$20x^2 - 109xy - 100y^2.$$

The positive factors of 20 and 100 are 2, 2, 5, 2, 5, 2 and 5.

wanted difference is 109

20 100 . . . . . difference is 80
16 125 . . . . . difference is 109

Note that 19, 18, and 17 were not candidates using the given factors. Resulting expression:

$$\begin{aligned} 20x^2 - 109xy - 100y^2 &= 20x^2 - 125xy + 16xy - 100y^2 \\ &= 5x(4x - 25y) + 4y(4x - 25y) \\ &= (5x + 4y)(4x - 25y) \end{aligned}$$

The next step is to consider those expressions whose third term is positive. The process is essentially the same except we are now looking for two numbers whose sum is the coefficient of the center term. The direction of your move is the same as it would have been, had you been considering a difference, i.e., if you desire a smaller sum, you move to the right from the smaller of the two numbers; if you desire a greater sum you move to the left from the smaller of the two numbers.

Consider this quadratic with the 3rd term positive:

$$6x^2 - 23x + 20.$$

Factors of 6 and 20 are 2, 3, 2, 2, 5. Since the third term is positive, you are looking for two numbers whose sum is 23.

wanted sum is 23

6 20 . . sum is 26: move right from 6
8 15 . . . . sum is 23

Moving to the right, 8 is the next number obtainable from the given factors. The product of the remaining factors is 15. Addition of these two numbers gives the desired sum of 23. Since the sign of the center term is negative, both 8 and 15 must be negative to give the value of -23. Resulting equation follows:

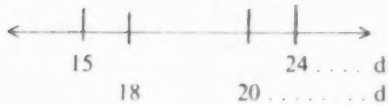
$$\begin{aligned} 6x^2 - 8x - 15x + 20 &= 2x(3x - 4) - 5(3x - 4) \\ &= (2x - 5)(3x - 4) \end{aligned}$$

Let's consider one last example to show that it makes no difference if the squared term were the 3rd term of the quadratic:

$$15 + 2x - 24x^2.$$

The factors of 15 and 24 are 3, 5, 2, 2, 2, 3.

wanted difference is 2



$$15 + 2x - 24x^2 = 15 - 18x + 20x - 24x^2$$
$$= 3(5 - 6x) + 4x(5 - 6x)$$
$$= (3 + 4x)(5 - 6x)$$

It has been my experience that even the low ability students can comprehend this method of factoring. The better students internalize it to the point that they can move down the number line mentally, giving them great speed in factoring any quadratic. I've also noted that many students become excited about algebra for the first time as they experience success in something they perceived to be difficult. For the instructor, it's nice to see your students enjoying algebra.

MATHEMATICAL TRIVIA (MT)

Edited by Ralph Selig
Fairleigh Dickinson University
Hackensack, NJ 07601

Below and in other places in this journal are some pieces of mathematical trivia that will hopefully challenge and entertain our readers.

The answers to the trivial questions appear at the end of the journal.

MT 1. What do each of the following computer-oriented abbreviations stand for?

a. BASIC
b. FORTRAN
c. COBOL

Numerical Integration: Surprises from the Trapezoidal Rule

by Travis Thompson
Box 934, Harding University, Searcy, Arkansas 72143

Travis Thompson received his Ph.D. in mathematics at the University of Arkansas. He has authored 20 papers ranging from general topology to pedagogical techniques. His hobbies are radio broadcasting and American cinema.

Introduction

Numerical integration techniques are assuming a greater role in the mathematics curriculum today because of the growth of computer science. Indeed, suggestions are now circulating in the mathematical community for consideration that more emphasis be placed on numerical integration and less emphasis on "techniques" of integration that produce closed form results.

However, the fact is that, while numerical integration is surprisingly rich in theory, computational results are often overlooked. In this article, some of these results will be presented along with an appropriate reference.

Assumption 1. All functions under consideration are "sufficiently" differentiable.

Assumption 2. All data points $\{x_0, x_1, \ldots, x_n\}$ used to subdivide an interval [a,b] are equally spaced;
i.e., $x_i - x_{i-1} = h(n) = (b-a)/n$, $1 \le i \le n$.

Assumption 3. The number of data points used to sub-divide an interval [a,b] is odd, and n = 2^m, for some integer m.

Trapezoidal Rule (2-Point Formula) And More

The Trapezoidal Rule is often called the 2-point rule since its development is based on two points. Assume that $a = x_0 < x_1 < x_2 < x_3 < x_4 = b$. Using $\{x_0, x_2, x_4\}$ and applying the Trapezoidal Rule produces

$$\int_a^b f(x)dx \approx [h(2)/2][f(x_0) + 2f(x_2) + f(x_4)].$$

Let T(2) denote this trapezoidal sum (the 2 in T(2) stands for the number of subdivisions of the interval [a,b] produced by the equally spaced points.) Similarly,

$T(1) = [h(1)/2][f(x_0) + f(x_4)]$, and

$T(4) = [h(4)/2][f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + f(x_4)]$.

Question #1: Why bother with the Trapezoidal Rule? Isn't Simpson's Rule (3-point rule) generally considered superior?

Reply: Perform this calculation: [4T(2) − T(1)] / 3. The result is $[h(2)/3][f(x_0) + 4f(x_2) + f(x_4)]$.

This is precisely Simpson's Rule for $\{x_0, x_2, x_4\}$!

Denote this by S(2); i.e., Simpson's Rule for 2 equal subdivisons of [a,b]. Try [4T(4) − T(2)] / 3 and you obtain S(4), Simpson's Rule for 4 equal subdivisions of [a,b].

[Note: This procedure is known as Richardson's Extrapolation which is used in Romberg Integration. The algorithm holds for any number of (2^m)+1 equally spaced points of the interval [a,b]. In terms of the notation already developed,

S(2^m) = [4T(2^m) − T(2^(m-1))] / 3.

Therefore, by calculating the trapezoidal sum, Simpson's sum becomes trivial to derive.]

Question #2: It's interesting that a combination of 2-point rule calculations yields a 3-point calculation. Is it possible to combine 3-point rule calculations to obtain a 5-point rule calculation?

Reply: Yes! The 5-point rule is

$$\int_a^b f(x)dx \approx [2h(4)/45][7f(x_0) + 32f(x_1) + 12f(x_2) + 32f(x_3) + 7f(x_4)].$$

To obtain the five-point rule from Simpson's Rule, simply calculate FIVE(4) = [16S(4) − S(2)] / 15, where FIVE(4) denotes the 5-point rule on an interval subdivided into four equal parts. The above coefficents of 7, 32, 12, 32, and 7 can be verified by carrying out this calculation algebraically.

The errors incurred by using the Trapezoidal and Simpson's Rules are functions of the second and fourth derivatives, respectively. Similarly, the 5-point rule incurs an error that is a function of the sixth derivative. Therefore, the five-point rule will produce exact answers when numerically integrating polynomials of degree five or less.

As an illustration, consider $\int_a^a (x^5 - x - 1)dx$ and use $x = i, 0 \le i \le 4$. All one needs to do to (numerically) integrate this polynomial is to find T(1), T(2), and T(4) which are 2036, 1076, and 756, respectively. Now set up a triangular array as follows:

T(1)

T(2) S(2) = [4T(2)-T(1)] / 3
= 756

T(4) S(4) = [4T(4)-T(2)] / 3
= 676

FIVE(4) = [16S(4)-S(2)] / 15
= 2012 / 3

The exact answer is 2012/3.

This procedure can be easily programmed in most programming languages. Also, the procedure can be extended to 9 (and higher)-point rules (See Hildebrand, 1974). A BASIC program can be found at the end of this article. The program was written to run on a VAX 11/750 (compiled VAX BASIC). The program will calculate trapezoidal sums, Simpson sums, five-point sums, and also nine-point sums and higher (the limit is machine dependent). For example, in running the program, if you enter a value of 5 when prompted (for 16 = 2^(5-1) subdivisions of the interval), the program will calculate the integral based

on a seventeen-point rule! Amazingly, this will integrate all polynomials of degree seventeen or less with (algebraically) no error!

This technique can produce awe in the student who is unfamiliar with it. One of the author's favorite tricks is to have his calculus class compute the trapezoidal sum using 2 points and then 3 points [T(1) & T(2)]. Afterwards, the students are asked to compute the Simpson's sum using three points. While the class toils, the author calmly calculates [4T(2)–T(1)]/3 and magically writes the answer on the board. At least for one day, the teacher is supreme in the eyes of his students!

Program

```
 5 REM--------------------The value of N should be modest (< = 8)
10 INPUT "Please enter N, where the number of subdivisions is 2^(N-1)";N
15 DIM R(N,N)
20 INPUT "Please enter the upper bound (B) and lower bound (A)";B,A
25 H = B - A
30 REM--------------------The following function is defined for
35 REM--------------------illustration. The user may define a
36 REM--------------------different function (line 40).
40 DEF FNF(X) = X^5 - X - 1
45 R(1,1) = 0.5*H*(FNF(A) + FNF(B))
50 PRINT R(1,1)
55 L = 1
60 FOR K = 2 TO N STEP 1
65       H = 0.5*H
70       L = L + L
75       SUM = 0.0
76 REM----------------Calculation begins for the Trapezoidal column
80     FOR J = 1 TO (L-1) STEP 2
85       X = A + J*H
90       SUM = SUM + FNF(X)
95     NEXT J
96 REM----------------Trapezoidal value loaded in row K, column 1
100   R(K,1) = 0.5*R(K-1,1) + H*SUM
105   M = 1
110     FOR J = 2 TO K STEP 1
115       M = 4*M
120       R(K,J) = R(K,J-1) + (R(K,J-1)-R(K-1, J-1))/ M-1)
125     NEXT J
126 REM----------------Row K of the Romberg Array is printed
130     FOR J = 1 TO K STEP 1
135       PRINT R(K,J);
140     NEXT J
145   PRINT
150 NEXT K
160 END
```

Reference

Hildebrand, F. B. (1974). *Introduction to Numerical Analysis.* New York: McGraw-Hill.

Bilateral Symmetry of the Hypergeometric Distributions

by Dr. Sandra M. Pulver
Department of Mathematics, Pace University
Pace Plaza, New York, NY 10038

Dr. Sandra M. Pulver is coordinator and host of the Mathematics Fair of New York, which holds its annual meeting at Pace University and judges outstanding high school talent in original mathematical work. She is an active member of the New York City and State Association of Mathematics Teachers. She was also Regional Chairperson of the International MAA Mathematics Contest, whose top scorers go on to compete in the Mathematical Olympiads. An associate professor at Pace University, she has taught mathematics for 22 years.

The binomial probability distribution is an elementary one and is covered in every standard introductory probability course. We use the binomial theorem when we have "n" independent trials and only two alternative outcomes for each trial, labeled a "success" or a "failure", and we wish to obtain the probability that an event will occur "x" times in "n" independent trials. Then the probability of "x" successes in "n" independent trials is:

$$P(x) = \binom{n}{x} p^x (1-p)^{n-x}$$

where $\binom{n}{x} = C(n,x)$ is the familiar combination formula, and p = the probability of one success.

A simple, yet intriguing question to pose to the class as they are working on their exercises is "Under what conditions will the binomial distribution be symmetric?" Students working through exercises will easily discover by themselves that the binomial distribution will be symmetric if $p = \frac{1}{2}$.

For example, when $p = \frac{1}{2}$ and x = 0, 1, 2, 3 and 4, $P(0) = \frac{1}{16}$, $P(1) = \frac{4}{16}$, $P(2) = \frac{6}{16}$, $P(3) = \frac{4}{16}$, $P(4) = \frac{1}{16}$ and the probabilities are symmetric. However, when p = .01 and x = 0,1,2,3,4, P(0) = .960596, P(1) = .0388119, P(2) = .00058806, P(3) = .00000396 and P(4) = .00000001 and the probabilities are not symmetric.

When the trials still retain their two alternative outcomes — "success" and "failure", but become dependent, as for example, sampling *without* replacement, the binomial theorem no longer applies. Consider the following problem:

> Among 16 stereos, 5 are defective. Eight stereos are randomly picked for inspection without replacement. What is the probability distribution for the number of stereos picked which are defective?

Since the probability of success changes from trial to trial, a new distribution, the hypergeometric distribution is applicable.

The hypergeometric distribution gives the probability of exactly "x" successes in "n" dependent trials:

$$P(x) = \frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}}$$

where a = the number of successes possible, b = the number of possible failures, n = the number of trials, and x = the number of successes desired.

If we call "picking a defective stereo" a "success", we may have 0 to 5 "successes", limited not by the number of trials, but by the number of possible successes. Hence, a = 5, b = 11, n = 8, and we have the following table:

Number of Successes *No. of Defective Stereos* x	*Probability* $p(x)$
0	$P(0) = \frac{\binom{5}{0}\binom{11}{8}}{\binom{16}{8}} = .013$
1	$P(1) = \frac{\binom{5}{1}\binom{11}{7}}{\binom{16}{8}} = .128$
2	$P(2) = \frac{\binom{5}{2}\binom{11}{6}}{\binom{16}{8}} = .359$
3	$P(3) = \frac{\binom{5}{3}\binom{11}{5}}{\binom{16}{8}} = .359$
4	$P(4) = \frac{\binom{5}{4}\binom{11}{4}}{\binom{16}{8}} = .128$
5	$P(5) = \frac{\binom{5}{5}\binom{11}{3}}{\binom{16}{8}} = .013$

Having already discovered the symmetry in the binomial distribution, students might be curious to see that this distribution is symmetric. It was certainly not due to a value of "p" since "p" does not even exist here; the probability of a success changes from trial to trial.

A very interesting extra or enrichment exercise to pose to students is the following question: Why is this distribution symmetric, or, more generally, *when* is the hypergeometric distribution symmetric?

By symmetry, here, we mean the condition that

$$P(x) = P(a-x) \qquad \text{for } x = 0,1,2,\ldots a.$$

Since denominators are always equal, we can concern ourselves only with numerators. Our symmetry condition will be satisfied if

$$\binom{a}{x}\binom{b}{n-x} = \binom{a}{a-x}\binom{b}{n-(a-x)} \qquad \text{for } 0 \le x \le a.$$

We know that $\binom{a}{r} = \binom{a}{s}$ iff $r+s = a$ (or $r = s$).

Since $x + (a-x) = a$, we have $\binom{a}{x} = \binom{a}{a-x}$. We only need

$$\binom{b}{n-x} = \binom{b}{n-(a-x)}.$$

This will be true when $(n-x) + n - (a-x) = b$, or $n = \frac{a+b}{2}$. (It is also true when $n-x = n-(a-x)$, or $x = \frac{a}{2}$, but this is simply the "middle" of the distribution when "a" is an even number; i.e., $P(x) = P\left(\frac{a}{2}\right) = P(a-x)$.)

Thus the hypergeometric distribution, when $a < n$, is symmetric if $n = \frac{a+b}{2}$, i.e., if the number of trials is equal to half the total of possible successes plus possible failures. Of course this means that the "population" size, a+b, must be an even number.

However, what happens in the case where $a \geq n$? Consider this problem:

> Among 16 stereos, 8 are defective. Four stereos are picked at random for inspection. What is the probability distribution for the number of defective stereos?

Here, n=4 and is less than "a", which equals 8. Our successes are therefore limited by the number of trials, n. We now have the following table:

Number of Successes *No. of Defective Stereos* x	*Probability* $P(x)$
0	$P(0) = \dfrac{\binom{8}{0}\binom{8}{4}}{\binom{16}{4}} = .038$
1	$P(1) = \dfrac{\binom{8}{1}\binom{8}{3}}{\binom{16}{4}} = .246$

2 $$P(2) = \frac{\binom{8}{2}\binom{8}{2}}{\binom{16}{4}} = .431$$

3 $$P(3) = \frac{\binom{8}{3}\binom{8}{1}}{\binom{16}{4}} = .246$$

4 $$P(4) = \frac{\binom{8}{4}\binom{8}{0}}{\binom{16}{4}} = .038$$

For a distribution such as this (in which the number of successes, x, are limited by the number of trials, n), to be symmetric, we must have

$$P(x) = P(n-x)$$

or:

$$\frac{\binom{a}{x}\binom{b}{n-x}}{\binom{a+b}{n}} = \frac{\binom{a}{n-x}\binom{b}{n-(n-x)}}{\binom{a+b}{n}}, \quad 0 \leq x \leq n,$$

i.e., $$\binom{a}{x}\binom{b}{n-x} = \binom{a}{n-x}\binom{b}{x}$$

This symmetry condition, $P(x) = P(n-x)$ for $x=0,1,\ldots,n$ will be satisfied if "a", the number of possible successes equals "b", the number of possible failures.

An exposition of the hypergeometric distribution in conjunction with the binomial distribution will serve to reinforce the concept of dependent and independent trials. In addition, the enrichment topic of symmetry in both distributions will provide a satisfying conclusion of the topic for more able students.

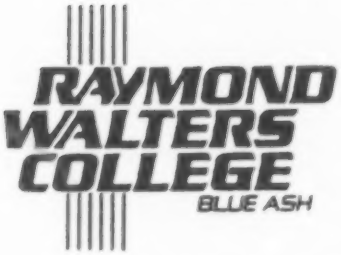
References

Brownlee, K.A. (1960). *Statistical Theory and Methodology in Science and Engineering*. New York: Wiley.

Feller, William. (1950). *An Introduction to Probability Theory and its Applications*. New York: Wiley.

A Recursive Algorithm for Solving Linear Diophantine Equations

by Gene P. Blanford
Raymond Walters College, Blue Ash, Ohio 45236



Gene Patrick Blanford is an Associate Professor of Math, Physics, and Computer Science at Raymond Walters College. His teaching assignment includes courses in Advanced BASIC, FORTRAN, MODULA-2, PASCAL, PL/1, Data Structures and VAX Assembly Language. His interests in computer science are in data structures and languages.

If the rules of a programming language allow a function or procedure to invoke itself, the language is said to allow recursion and any function or procedure calling itself is said to be recursive. Some tasks or problems are inherently recursive in nature and recursive algorithms to do the task or find solutions to the problem may only require a few lines of code when translated to a high level programming language. Recursion is usually not the most efficient way in terms of computer time and space to do a task or find a solution to a problem, but it may save a lot of programming time. In this paper a simple recursive algorithm is used to obtain a solution to a linear Diophantine equation.

A Diophantine equation is a rational integral algebraic equation in which the coefficients of the variables, the constant term, and the values of the variables which satisfy the equation must all be integers. The linear Diophantine equation is of the form:

I. $Ax + By = C$

The method given does not deal with equations in which both A and B are equal to zero. Their solutions are obvious.

The method of finding a solution to a linear Diophantine equation that I offer here consists of two steps. The first step is to use the Euclidean algorithm to find d where d is the greatest common divisor of $|A|$ and $|B|$. If d is not a factor of C, then there are no solutions. If d is a factor of C, then we construct an equivalent equation by dividing A, B, and C by d.

The second step begins by checking if $A = 0$. If $A = 0$, a value of zero is returned for X and we proceed to find a value of $Y = C / B$, $B <> 0$. The case of $A = B = 0$ is not considered. If $A <> 0$, the equation is rearranged in the following manner:

$Ax + [A(nl) + (Rbl)]y = A(ml) + (Rcl)$, where $B \bmod A = (Rbl)$ giving $B = A(nl) + (Rbl)$ and
$C \bmod A = (Rcl)$ giving $C = A(ml) + (Rcl)$

$(Rbl)y + A[x + (nl)(y) - (ml)] = (Rcl)$
$(Rbl)y + Az = (Rcl)$, where $z = x + (nl)(y) - (ml)$,
giving a new linear Diophantine equation.

II. $(A1)y + (B1)z = (C1)$. $\quad |A| > |A1|$

If C1 mod A1 = 0, let z = 0 which will give a solution for y (y = C1/A1) which can be substituted into (I) to find x. If C1 mod A1 is unequal to 0 the above steps may be repeated to give the following:

III. (A2)z + (B2)w = (C2), $|A| > |A1| > |A2|$

If C2 mod A2 = 0, let w = 0 which will give a solution for z (z = C2/A2) which can be substituted into (II) to find y which can be substituted into (I) to find x.

It is obvious that the above process only need be repeated a finite number of times until Ci mod Ai = 0, i = 1,2. . . n.

Applying the above method to 342x - 106y = 422 we would obtain:

Step 1. 342x - 106y = 422 GCD(342,106) = 2

Step 2.
171x - 53y = 211
-53y + 171z = 40
12z - 53s = 40
-5s + 12t = 4
2t - 5v = 4

Since 4 mod 2 = 0, we let v = 0 which $\Rightarrow$ t = 2 $\Rightarrow$ s = 4 $\Rightarrow$ z = 21
$\Rightarrow$ y = 67 $\Rightarrow$ x = 22.

A simple recursive algorithm to do the above is as follows:

```
FUNCTION Reduce (A, B, C :INTEGER){A, B, and C passed by value.}
    IF(A<>0) THEN
      IF((C MOD A) = 0) THEN
        RETURN (C/A)
      ELSE
        RETURN ((C - B*(Reduce((B Mod A), A, (C MOD A))))/A)
    ELSE
        RETURN (0)
END Reduce
```

The calling procedure would access the function by

X = Reduce (A, B, C)

This function may easily be implemented in many high level programming languages such as Pascal, Modula-2, PL/1, C, or some of the newer Basics.

The program that follows is a somewhat, in the interest of programming economy, terse but working Turbo Pascal program that utilizes the recursive algorithm given above. If solutions exist, the program only provides a single solution (X,Y) for each set of values entered. Other solutions (x,y) may be produced by the equations x = (X + K * B) and y = (Y - K * A) for K an integer. The reader may find it of interest to add an output statement listing the contents of A, B, and C as the first statement in the function Reduce.

Written by Gene P. Blanford. This is a no frills program to illustrate the use of recursion to find a solution for any linear Diophantine equation of the form Ax + By = C.

```
PROGRAM Diophantine (INPUT, OUTPUT);
  VAR              X, Y, CoefA, CoefB, ConstC, GCD  :INTEGER;
```

```
FUNCTION      Reduce (A, B, C  :INTEGER): INTEGER;  BEGIN
  IF(A<>0) THEN
    IF((C MOD A) = 0) THEN
```

```
      Reduce := C DIV A
    ELSE
      Reduce := (C - B * Reduce (B MOD A, A, C MOD A)) DIV A
  ELSE Reduce := 0
END; (* reduce *)
```

```
FUNCTION     EuclidAlgor (Smaller, Larger  :INTEGER)  :INTEGER;
  VAR        Remainder   :INTEGER;
BEGIN
  WHILE (Smaller<>0) DO BEGIN
    Remainder := Larger mod Smaller;
    Larger := Smaller;
    Smaller := Remainder;
  END; (* do while *)
  EuclidAlgor := Larger
END; (* euclidean algorithm *)
```

```
PROCEDURE  GetCoeficients (VAR CoefA, CoefB, ConstC :INTEGER); BEGIN
  WRITELN(' This program will find a solution for a Diophantine');
  WRITELN(' equation of the form Ax + By = C, where A, B and C are');
  WRITELN(' integers.'); WRITELN; WRITELN;
  WRITE(' Enter, separated by a space, values for A, B and C. ==>');
  READLN( CoefA, CoefB, ConstC); WRITELN; WRITELN;
  WRITELN (':5, CoefA, 'X + (', CoefB:0,') Y = ', ConstC:0); WRITELN;
  IF (CoefA < 0) THEN BEGIN
    CoefA := - CoefA; CoefB := -CoefB; ConstC := -ConstC
  END (* if *)
END; (* get coeficients *)
```

```
BEGIN (* main program *)
  GetCoeficients (CoefA, CoefB, ConstC);
  If (ABS (CoefA)<= ABS (CoefB)) THEN
    GCD := EuclidAlgor (ABS(CoefA), ABS(CoefB))
  ELSE
    GCD := EuclidAlgor (ABS(CoefB), ABS(CoefA));
  IF (GCD = 0) THEN
    IF (ConstC = 0) THEN
      WRITELN(':5, 'Solution is (x,y); x and y any two integers.')
    ELSE
      WRITELN (':5, 'No solution exists.')
  ELSE BEGIN
    IF((ConstC MOD GCD)<>0) THEN
      WRITELN ('No solution exists since GCD(a,b) is not a divisor of c.')
  ELSE BEGIN
        IF(GCD  > 1) THEN BEGIN
          CoefA := CoefA DIV GCD; CoefB := CoefB DIV GCD;
          ConstC := ConstC DIV GCD
        END; {if}
        X := Reduce (CoefA, CoefB, ConstC);
```

```
        IF (CoefB<>0) THEN  Y := (ConstC - CoefA * X) DIV CoefB
        ELSE Y := 0;
        WRITELN(OUTPUT,'':5,  'A solution is (',X,',',Y,').')
    END (* else *)
  END (* else *)
END.  (* main program *)
```


CONTENT YOU CAN COUNT ON— IN A PRESENTATION THAT WORKS.

College Algebra

386 pages • hardcover • Study Guide • Instructor's Edition • Alternate Testing Program • Computerized Test Generator • Solutions Manual • GPA: Grade Performance Analyzer • 1986

College Trigonometry

276 pages • hardcover • Instructor's Edition • Alternate Testing Program Computerized Test Generator • Solutions Manual • GPA: Grade Performance Analyzer • 1987

College Algebra and Trigonometry

532 pages • hardcover • Ancillaries as for *College Algebra* • 1987

All by **Timothy J. Kelly,** Hamilton College
Richard H. Balomenos, University of New Hampshire
John T. Anderson, Hamilton College

The Kelly/Balomenos/Anderson program—now available in two separate texts or in a single volume—offers class-tested, authoritative mathematical content and a clear, goal-referenced organization.

For adoption consideration, request examination packages from your regional Houghton Mifflin office.

Houghton Mifflin

13400 Midway Rd., Dallas, TX 75244-5165 1900 S. Batavia Ave., Geneva, IL 60134
925 E. Meadow Dr., Palo Alto, CA 94303 University Sq., 101 Campus Dr., Princeton, NJ 08540

An Induction Proof of the Inequality of the Weighted Geometric and Arithmetic Means

By Terence Brenner

Eugenio Maria De Hostos Community College of the
City University of New York
500 Grand Concourse
Bronx, New York 10451

Terence received his B.A. and M.A. in mathematics from Herbert H. Lehman College of the City University of New York. He received his Ph.D. from Yeshiva University in 1984. The title of his dissertation is "On the Spectra of the Schrodinger Operator."

Introduction

The inequality of the geometric and arithmetic means states that for n positive rational numbers $a_1, a_2, \ldots, a_n$,

$$a_1 a_2 \ldots a_n \leq \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n \qquad (1)$$

and equality holds if and only if $a_1=a_2=\ldots=a_n$. For example, given the numbers 4 and 8

$$(4)(8) \leq \left(\frac{4+8}{2}\right)^2.$$

We can also include weights in the inequality. The weighted form, the inequality of the weighted geometric and arithmetic means, states that for n positive rational numbers $a_1, a_2, \ldots, a_n$ and n positive rational weights $P_1, P_2, \ldots, P_n$

$$a_1^{P_1} a_2^{P_2} \ldots a_n^{P_n} \leq \left(\frac{P_1a_1+P_2a_2+\ldots+P_na_n}{P_1+P_2+\ldots+P_n}\right)^{P_1+P_2+\ldots+P_n} \qquad (2)$$

and equality holds if and only if $a_1=a_2=\ldots=a_n$. For example, given the numbers 4 and 16 and their respective weights .5 and .25

$$(4)^{.5}(16)^{.25} \leq \left(\frac{.5(4)+.25(16)}{.5+.25}\right)^{.5+.25}.$$

Hardy, Littlewood and Polya (1964, pp. 16-21) cite several proofs of these inequalities. For example, Cauchy used backward induction and Chrystal used ordinary induction and the theorem that states that if x is positive and $\neq 1$, then $nx^{n-1}(x-1) > x^n-1 > n(x-1)$ unless n lies between 0 and 1, in which case $nx^{n-1}(x-1) < x^n-1 < n(x-1)$. In this paper we use ordinary induction to prove the unweighted form of the inequality of the geometric and arithmetic means. We will use the binomial expansion and the inequality theorem that

states that for real numbers A,B,C and D, if $C \leq D$ and $A-C < B-D$, then $A < B$. We will then use the unweighted form of the inequality of the geometric and arithmetic means to directly prove the weighted form of the inequality of the geometric and arithmetic means.

The Proofs

The proof of the inequality of the geometric and arithmetic means:

$$P_n : a_1 a_2 \ldots a_n \leq \left(\frac{a_1+a_2+\ldots+a_n}{n}\right)^n \tag{3}$$

To prove by induction we must show that P_1 is true and if P_k is true, then P_{k+1} is true.

1) P_1 is true, since $a_1 \leq \left(\frac{a_1}{1}\right)^1$.

2) If $a_1 a_2 \ldots a_k \leq \left(\frac{a_1+a_2+\ldots+a_k}{k}\right)^k$, (4)

then $a_1 a_2 \ldots a_{k+1} \leq \left(\frac{a_1+a_2+\ldots+a_k+1}{k+1}\right)^{k+1}$. (5)

Can you rearrange the sequence $a_1, a_2, \ldots, a_{k+1}$ such that

$$\frac{a_1+a_2+\ldots+a_k}{k} < a_{k+1} \quad ? \tag{6}$$

We shall show if (6) is not possible, then we have equality and if (6) is possible, we have the inequality.

Case 1: (6) is not possible.

We assume all the a_i's are not equal and then show there is a contradiction. Rearrange the sequence such that

$$a_{k+1} = \max\{a_1, a_2, \ldots, a_{k+1}\} \tag{7}$$

then

$$\frac{a_1+a_2+\ldots+a_k}{k} < a_{k+1}. \tag{8}$$

But (8) is (6), hence the contradiction. We conclude that $a_1 = a_2 = \ldots = a_{k+1}$ and equality holds.

Case 2: (6) is possible.

All the a_i's are not equal. We will show that

$$a_1a_2 \ldots a_{k+1} < \left(\frac{a_1+a_2+\ldots+a_{k+1}}{k+1}\right)^{k+1}. \tag{9}$$

Let d be the average of $a_1, a_2, \ldots, a_k$. Then we have:

$$\text{i)} \quad \frac{a_1+a_2+\ldots+a_k}{k} = \frac{a_1+a_2+\ldots+a_k+d}{k+1} \tag{10}$$

$$\text{ii)} \quad d = \frac{a_1+a_2+\ldots+a_k+d}{k+1} \tag{11}$$

iii) We know that $a_{k+1} > d$. Then there exists a positive integer e and a rational number r such that

$$a_{k+1} = e\,d + r, \text{ where } e \geq 1 \text{ and } 0 \leq r < d. \tag{12}$$

To prove (9) we will show that

$$a_1a_2 \ldots a_k d \leq \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1} \tag{13}$$

and

$$a_1a_2 \ldots a_{k+1} - a_1a_2 \ldots a_k d < \left(\frac{a_1+a_2+\ldots+a_{k+1}}{k+1}\right)^{k+1} - \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1}. \tag{14}$$

1) First we prove (13).

Using the induction hypothesis, we have

$$a_1a_2 \ldots a_k \leq \left(\frac{a_1+a_2+\ldots+a_k}{k}\right)^k. \tag{15}$$

We multiply both sides of (15) by d ($d > 0$) to obtain

$$a_1a_2 \ldots a_k d \leq \left(\frac{a_1+a_2+\ldots+a_k}{k}\right)^k d. \tag{16}$$

Using (10) we have

$$= \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^k d. \tag{17}$$

Now use (11) to obtain

$$= \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^k \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right). \tag{18}$$

Hence,

$$a_1a_2 \ldots a_k d \leq \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1}. \tag{19}$$

2. Next we prove (14).

We start with

$$\left(\frac{a_1+a_2+\ldots+a_{k+1}}{k+1}\right)^{k+1} - \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1}. \tag{20}$$

Using (12) we have

$$= \left(\frac{a_1+a_2+\ldots+a_k+(ed+r)}{k+1}\right)^{k+1} - \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1}. \tag{21}$$

Now we use the fact that ed+r = d+(e-1)d+r to obtain

$$= \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1} + \frac{(e-1)d+r}{k+1}\right)^{k+1} - \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1}. \tag{22}$$

Applying the binomial expansion to (22), we look at the first two terms and define u to be the rest of the expansion. Since all numbers that we use in (22) are positive, u>0. Therefore, the expression on line 20 equals

$$\left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1} + (k+1)\left(\frac{(e-1)d+r}{k+1}\right)\left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k} + u \quad \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1}. \tag{23}$$

Combining and simplifying, we have

$$= [(e-1)d+r]\left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k} + u. \tag{24}$$

Using (10) and u >0 we have

$$> [(e-1)d+r]\left(\frac{a_1+a_2+\ldots+a_k}{k}\right)^{k}. \tag{25}$$

Now use the fact that (e-1)d+r >0 and (15) to obtain

$$\geq [(e-1)d+r]\, a_1a_2\ldots a_k\,. \tag{26}$$

We have just proved that

$$[(e-1)d+r]\, a_1a_2\ldots a_k < \left(\frac{a_1+a_2+\ldots+a_{k+1}}{k+1}\right)^{k+1} - \left(\frac{a_1+a_2+\ldots+a_k+d}{k+1}\right)^{k+1}. \tag{27}$$

In order to complete the proof we need to show that

$$a_1a_2\ldots a_{k+1} - a_1a_2\ldots a_kd = [(e-1)d+r]\, a_1a_2\ldots a_k\,. \tag{28}$$

We start with

$$a_1a_2 \ldots a_{k+1} - a_1a_2 \ldots a_kd \,. \tag{29}$$

Using (12) we have

$$= a_1a_2 \ldots a_k\,(ed+r) - a_1a_2 \ldots a_kd \,. \tag{30}$$

Now use the fact that $ed+r = d+(e-1)d+r$ to obtain

$$= a_1a_2 \ldots a_k[d+(e-1)d+r] - a_1a_2 \ldots a_kd \,. \tag{31}$$

Distributing, we obtain

$$= a_1a_2 \ldots a_kd + a_1a_2 \ldots a_k[(e-1)d+r] - a_1a_2 \ldots a_kd \,. \tag{32}$$

Simplifying, we have

$$= a_1a_2 \ldots a_k[(e-1)d+r] \,. \tag{33}$$

We have just showed that (14) is true and the proof is now complete.

The proof of the inequality of the weighted geometric and arithmetic means:

$$\text{Define } \frac{v}{w} = P_1+P_2+ \ldots +P_n \text{, where v and w are positive integers.} \tag{34}$$

Multiply both sides of (34) by w getting

$$v = w\,P_1 + w\,P_2 + \ldots + w\,P_n \,.$$

Now we restate (2) using (34)

$$a_1^{P_1}\,a_2^{P_2} \ldots a_n^{P_n} \le \left(\frac{P_1a_1+P_2a_2+ \ldots +P_na_n}{v/w}\right)^{v/w} . \tag{36}$$

Inequality (36) is true if and only if

$$a_1^{w\,P_1}\,a_2^{w\,P_2} \ldots a_n^{w\,P_n} \le \left(\frac{w\,P_1a_1 + w\,P_2a_2 + \ldots + w\,P_na_n}{v}\right)^{v} \tag{37}$$

is true.

To complete the proof we must prove (37). Since w is a positive integer, $w\,P_1$, $w\,P_2$, . . . , $w\,P_n$ are positive integers.

We start with

$$a_1^{w P_1} a_2^{w P_2} \cdots a_n^{w P_n} = \overbrace{a_1 \cdots a_1}^{w P_1 \text{ factors}} \cdot \overbrace{a_2 \cdots a_2}^{w P_2 \text{ factors}} \cdot \cdots \cdot \overbrace{a_n \cdots a_n}^{w P_n \text{ factors}} \cdot \tag{38}$$

Using the inequality of the geometric and arithmetic means on (38) we have

$$\leq \left(\frac{\overbrace{a_1 + \cdots + a_1}^{w P_1 \text{ terms}} + \overbrace{a_2 + \cdots + a_2}^{w P_2 \text{ terms}} + \cdots + \overbrace{a_n + \cdots + a_n}^{w P_n \text{ terms}}}{w P_1 + w P_2 + \cdots + w P_n} \right)^{w P_1 + w P_2 + \cdots + w P_n} \tag{39}$$

Using (35) we obtain

$$= \left(\frac{w P_1 a_1 + w P_2 a_2 + \cdots + w P_n a_n}{v} \right)^v . \tag{40}$$

The proof is now complete.

REFERENCES

Hardy, G. E., Littlewood, J. E., & Polya, G. (1964). *Inequalities*. London: Cambridge at the University Press.

MT 2. What is the next number?
77, 49, 36, 18, _____

The Derivative of Sin(kx) — Why the Chain Rule?

by Ann D. Holley
San Diego Mesa College
Mesa College Drive
San Diego, CA 92111

THE SAN DIEGO COMMUNITY COLLEGE DISTRICT

Ann Holley has been a Mathematics Instructor in the San Diego Community Colleges since 1971 when she received the M.S. in Mathematics from San Diego State University. Prior to that, she taught at both secondary and four-year college levels. Her half a dozen published articles reflect her strong interest in finding better ways to reach into students' thought processes.

The need for the Chain Rule in finding the derivative of sin(kx) is often a sticking point for beginning calculus students — particularly when the derivatives of the trigonometric functions are studied immediately after the introduction of the Chain Rule. I have usually managed to convince students with an argument which goes somewhat as follows:

> "After all, $D_x[\sin(5x^2+7x)] = [\cos(5x^2+7x)](10x+7)$. The Chain Rule is clearly needed here. $D_x\sin[g(x)] = \cos[g(x)]\ D_x g(x)$. If $g(x) = 6x$, then $D_x\sin(6x) = [\cos(6x)]\ D_x(6x) = 6\cos(6x)$."

A much better explanation was given to me by a very thoughtful beginning calculus student, Jeff Monroe, when he first encountered the problem. Jeff reasoned from the graph of sin(6x). Since the period is $\pi/3$, the graph must rise and fall much more rapidly than does the graph of the sine function. If we did not use the Chain Rule, the derivative of sin(6x) would be cos(6x). The maximum of the absolute value of the slope of the graph would be only 1 — not large enough, since the graph is obviously much steeper than the graph of sine. "It wouldn't get up there fast enough", he said (see figure 1 for a proof by picture). Similarly, $|D_x\sin(x/6)| = (1/6)|\cos(x/6)| \leq 1/6$. The graph of sin(x/6) rises and falls more slowly than does the graph of sine (x).

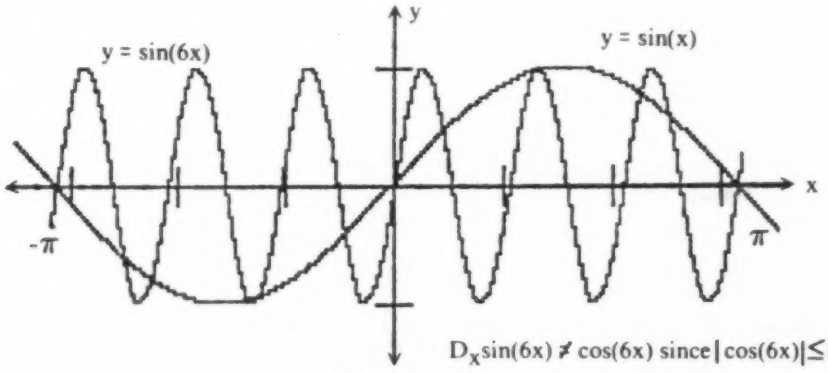
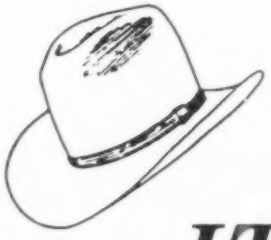


Figure 1

One can generalize the astute student's observations. It is well known and frequently taught that if $g(x) = f(kx)$ and $k > 1$, that the graph of g is a horizontal compression of the graph of f. If $0<k<1$, then the graph of g is a horizontal stretch of the graph of f. We can extend this to point out that if f is also differentiable, corresponding changes **must** occur in the graph of the slope function. If $k>1$, then the graph of the derivative of g, that is, $k(f'(kx))$, is a horizontal compression **and** a vertical stretch of the graph of $f'(x)$ --obviously the necessary alteration in the slope function. In case $0<k<1$, then the graph of the derivative of g, $k(f'(kx))$, is both a horizontal stretch **and** a vertical compression of the derivative of f. Thus in the elementary case where $g(x) = f(kx)$ and f is diffentiable we have an easily accessible geometric interpretation of the Chain Rule. We also have a convincing geometric argument that the derivative of $\sin(kx)$ is not simply $\cos(kx)$, but $k\cos(kx)$.



IT'S COMING!

1988 AMATYC CONVENTION

Calgary
Alberta, Canada
Oct. 27th — Oct. 30th, 1988

ACROMATYCS

(ACROnyms for Mathematical Associations of Two Year Colleges)

by Joseph Browne
Mathematics Department, Onondaga Community College
Syracuse, NY 13215

Joe has been teaching at the college level since 1969. He is presently the problem editor for the ***Mathematics and Computer Education Journal*** *and is a former president of the New York State Mathematics Association of Two Year Colleges.*

I belong to an organization with a rather peculiar name: NYSMATYC. Readers of this journal realize, of course, that NYSMATYC is not the actual title; it's an acronym for New York State Mathematics Association of Two Year Colleges, and it's pronounced niz-MAA-tik (except in a couple areas of the state where the dialect changes it to NICE-maa-tik). In New England the analogous organization is NEMATYC, pronounced KNEE-maa-tik, and the Pennsylvania State version is PSMATYC, pronounced very carefully.

The creation of names for new organizations, procedures, and equipment requires a skill which is, perhaps, not fully appreciated by the general population. These names must often be quite long in order to properly specify the intended scope so that people hearing or reading the name will know whether or not they are interested. Clarity is certainly a virtue, but the mere length of the names becomes an albatross for anyone who must repeat them often. How many times a day would you want to say "National Aeronautical and Space Administration," for example? The result is that we often resort to using just the initials, and in speech, this is even further shortened if the initials form something we can pronounce, an acronym. The organization mentioned above is probably better known by its acronym, NASA, than by its actual name. The art of creating names and titles has now advanced to the point where the formation of an acronym is as important as the descriptiveness of the title. One source even recommends choosing the acronym first and then the title. This yields a one-to-many mapping which is easier to work with than the other way around.

Acronyms are not all created equal, however, and for convenience I have divided them into three classes. Class I contains sets of initials (and word fragments) which are pronounceable, but do not form an existing word. NASA fits in this category, as do these other familiar ones: ARCO, UNICEF, FORTRAN, NORAD, and NATO. Members of the Society for the Preservation and Encouragement of Barber Shop Quartet Singing in America think you can pronounce SPEBSQSA, so I suppose that goes in Class I, too. It is worth noting that a few acronyms have escaped from this class and become accepted as genuine words. Radar (Radio Detection and Ranging), SCUBA (Self-Contained Underwater Breathing Apparatus), and SNAFU (Situation Normal, All F. . . . . Up) have entered the language, and LASER (Light Amplification by Stimulated Emission of Radiation) seems about ready to make the move.

Class II acronyms are known words, but words which have nothing to do with the title they are extracted from. One example which is often in the news is the SALT negotiations. Others seen less often are WHO (World Health Organization), the early computer MANIAC (Mathematical Analyzer, Numerical Integrater, and Computer), and ACORN (Automatic Checkout and Recording Equipment). Some times acronyms of this sort can be rather confusing. Herb Keller, formerly head of the Society for Industrial and Applied Mathematics, once attended a White House reception for leaders of the scientific community. He reports that then-president Gerald Ford had quite a puzzled look on his face when he was introduced to the "President of SIAM."

The third group, Class III, represents acronyming at its highest level. Here, the word formed is indeed related to the matter being titled. Mutual of New York (MONY) emphasizes this connection in its advertising. Some other examples are the easy to learn computer language, BASIC (Beginners All-purpose Symbolic Instruction Code), SAVES (Southern Area Volunteer Emergency Service), and OASIS (Organization for the Advancement of Space Industrialization and Settlement). The U.S. Naval Ordnance Laboratory calls Deep Underwater Nuclear Counting DUNC, and Raytheon's Homing All the Way Killer missile is known as HAWK. Educational computer users may be familiar with PLATO (Programmed Logic for Automatic Teaching Operations) and one wonders what McGill University had in mind for its High Altitude Research Program, HARP. A couple honorable mentions in this class go to EGAD (Electronegative Gas Detector) and ORACLE (Oak Ridge Automatic Computer and Logical Engine). Some may question its inclusion, but ZIP Code (Zoning Improvement Plan) at least had the right intentions.

Most existing acronyms fall into Class I. This is regrettable since, with a bit of effort and imagination, we can often form more interesting Class II or III acronyms. For some illustrative examples, consider the two year college mathematical associations mentioned in the first paragraph. In Arizona the organization would be AZMATYC, but if it combines with California it could be CARIZMATYC. The New Jersey group could be called Garden MATYC, though there might be some pressure for NewMATYC. HydroMATYC would be appropriate for Minnesota, and in Michigan it would be AutoMATYC (much better than MICHMATYC, though there ought to be some way to use the letters of Hamtramck).

In addition to these regional groups, there will be some special interest subgroups of mathematics associations of two year colleges. Those interested in promoting practical algebra and geometry should form PRAGMATYC. Aspiring mathematical writers should join TheMATYC, and those who like puzzles and problem solving, EnigMATYC. The computer oriented crowd would join SysteMATYC. Mathematics teachers with interests in other field need organizations, too. Those who also teach optics must form PrisMATYC, and those with a biological leaning will want EnzyMATYC. Those who object can form SchisMATYC, those who lose interest, AnticliMATYC, and those who can't cope, PsychosoMATYC. Professors who can't get tenure and bounce from campus to campus can join NoMATYC, and old timers who now have trouble holding onto the chalk can commiserate with other members of RheuMATYC.

There could also be organizations for two year college mathematics teachers related to their interests beyond the campus boundries. Those hardy souls willing to hurl their bodies down a snowy hill could form SkiMATYC. The thespians should have DraMATYC, the filmgoers CineMATYC, and the coin collectors NumisMATYC. The medically inclined need SymptoMATYC. For international travelers, DiploMATYC would be

appropriate, and for those civil war buffs who traipse about old battlefields I propose AppoMATYCS.

My friends who teach English (if I still have any) tell me that I must stop this barrage of acronyms. They say it is harmful to our students, teaching them bad grammar and ridiculous spelling. Reluctantly, I have agreed to give up my insidious habit and join them in forming a new organization to promote the upgrading of basic grammar and mathematics in the two year colleges. Our name? Of course. GRAMMATYC!

Editor's Note: *Editorial Panel member Gordon Hoagland suggested that "washed up" authors might become members of LAUNDROMATYCS. If you have some cute acronyms that you would like to suggest, send them to our humorist Les Tanner.*

CALL FOR REFEREES

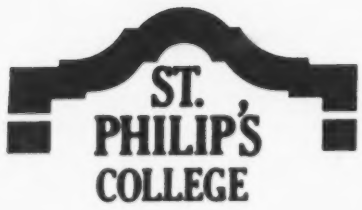
Readers are invited to help the editor to evaluate manuscripts by becoming members of *The AMATYC Review* reviewer panel. Write to Don Cohen, SUNY Ag-Tech College, Cobleskill, NY 12043 for more information.

Pre- and Post-Instructional Strategies for the College Developmental Mathematics Class

by Joannis Kelly Flatley
St. Philip's College

Dennis J. Gittinger
St. Philip's College

Adele Ducharme
Troy State University



Joannis Kelly Flatley is the Director of Educational Support Services at St. Philip's College in San Antonio, Texas. She has published several articles in the field of reading education and has a particular interest in adult literacy.

Dennis J. Gittinger is an assistant professor of mathematics at St. Philip's College in San Antonio, Texas. He is the author of several articles on mathematics and mathematics education.

Adele Ducharme is an assistant professor of education and chair of the elementary education department at Troy State University in Montgomery, Alabama. She has authored several articles in the field of reading education.

In junior and community colleges, many students function below the college level in both mathematics and reading skills. Mathematics instructors find themselves with classes where reading abilities range from second grade to college level. This range also implies differences in learning styles, interest, and prior knowledge about the content area. Because mathematics involves specialized reading skills, it is important to improve the reading comprehension and vocabulary abilities of these developmental students. The purpose of this article is to introduce to the mathematics instructor four strategies that can be used to introduce, teach, and reinforce concepts in their content area. These strategies are used in both the pre-instruction and post-instruction phases.

Pre-instruction Strategies

Three pre-instruction strategies, the anticipation/reaction guide (Bean and Peterson, 1981), the graphic organizer (Ausubel, 1960), and list-group-label (Moore, Readence, and Rickleman, 1982) will be discussed. Because students often have misconceptions about a topic (Bean & Peterson, 1981) and have varying degrees of knowledge about a topic, the pre-reading phase of instruction is vital for successful learning. Before students can effectively begin bridging existing knowledge with new information, misconceptions must

be identified and clarified. The anticipation/reaction guide provides the teacher with an informal diagnosis of students' conceptions of a topic while simultaneously activating the students' prior knowledge about this topic. The procedure for using the guide is to present students with four to six statements that require reflection on a topic.

The teacher may put the guide on a transparency or on ditto sheets for each student. As the teacher reads each statement aloud, students are to check on the "anticipation" side those statements with which they agree. Discussion about the checked statements follows which allows the teacher to note misconceptions and allows students to activate their prior knowledge. The guide statements can then be used to set purposes for reading, or the guide can be put away until after the reading phase is completed.

ANTICIPATION/REACTION GUIDE

PERCENTS

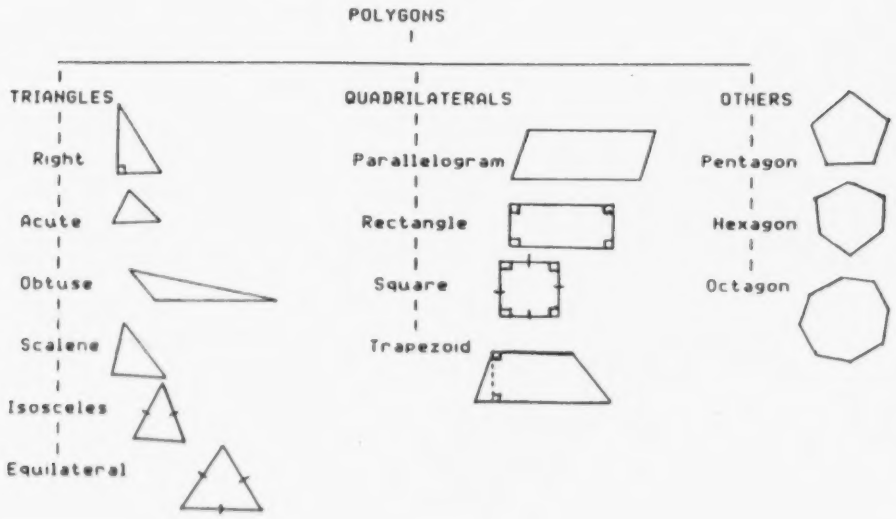
Directions: Under "anticipation," put an X before the statements with which you agree. At the end of the unit, you will be asked to re-mark this guide.

ANTICIPATION	REACTION	STATEMENT
_____	_____	1. Percent is a special ratio that is used to compare numbers to 100.
_____	_____	2. Percents, decimals, and fractions are basically the same thing.
_____	_____	3. Percents are rarely used in everyday life.
_____	_____	4. 100% is the largest percent possible.
_____	_____	5. Percents are used only to compute grades and to compute interest rates in savings accounts.

Another strategy which could be used in the pre-reading phase — either instead of the guide or along with it — is a graphic organizer. This strategy introduces the concepts and vocabulary which are found in the reading selection. This technique is very effective because (1) it provides a visual connection between the important concepts and words, (2) it organizes concepts hierarchically, (3) it activates students' prior knowledge of concepts and vocabulary, and (4) it helps students to focus on relevant information.

By graphically displaying the concepts and vocabulary, students can see the interrelationships between the two. The teacher must explain the organizer and how it works to assure maximum comprehension. Each concept is discussed and each term defined. Students are encouraged to ask questions or to contribute their knowledge about any of the ideas presented. Often, some students are very knowledgeable about a concept and can add more information to the organizer. These additions can help clarify the concepts for the other students. After all additions have been made, the teacher can then either display one large graphic organizer or give each student a copy of this organizer for reference as the students read the text. The organizer can help students focus on the relevant information.

GRAPHIC ORGANIZER



The third pre-instruction strategy, the list-group-label technique, is a strategy that uses categorization as a way to help students organize concepts. This strategy is most effective when much of the vocabulary is familiar to the students. When used as a pre-instruction strategy, the list-group-label lesson activates prior knowledge and enhances comprehension. With this strategy, the students brainstorm to produce a list of words that they associate with a topic. After the list has been generated, the students group the words and assign labels to each group. If appropriate, a graphic organizer can be constructed.

LIST — GROUP — LABEL

STIMULUS TOPIC: Measurement

Yard	Millimeter	Foot	Mile	Fahrenheit
Inch	Celsius	Acre	Ruler	Square Inch
Yardstick	Ounce	Quart	Pound	Square Yard
Gallon	Gram	Ton	Kilogram	Square Mile
Centimeter	Cup	Pint	Kilometer	Cubic Inch
Cubic Yard				

CATEGORIES

UNITS OF LENGTH

Inch
Foot
Yard
Mile
Kilometer
Centimeter
Millimeter

MEASUREMENT TOOLS

Ruler
Yardstick

UNITS OF AREA

Acre
Square Inch
Square Yard
Square Mile

UNITS OF VOLUME

Cubic Inch
Cubic Yard

UNITS OF WEIGHT

Ounce
Pound
Gram
Kilogram
Ton

UNITS OF LIQUID MEASURE

Cup
Pint
Quart
Gallon

UNITS OF TEMPERATURE

Fahrenheit
Celsius

By using pre-instruction strategies like the anticipation/reaction guide, the graphic organizer, and list-group-label, students are able to activate their prior knowledge about a topic and begin enhancing the integration of old information with the new.

Post-instruction Strategies

After the text has been read and the lectures given, it is imperative to provide some activities that will reinforce the new vocabulary and that will require application of the concepts that have just been learned. Developmental mathematics students profit from review activities that help make the new information less abstract and that help bring closure to a topic. Clues and questions is a strategy that is designed to help students review technical vocabulary and reinforce newly learned concepts. This strategy proceeds as follows: (1) Topic-related vocabulary words are collected and the words are typed on notecards, one word per card. (2) The students' task is to write a question — on the other side of each card — whose answer is the word on the card. (3) Students then divide into groups with each group having a portion of the cards. One student shows the question side of a card to the other members of the group. The other students in the group take turns asking questions until the word is identified. The procedure is continued until all of the cards have been used.

By participating in a fun activity like clues and questions, students are able to review lecture and text materials in a pleasant and non-threatening manner.

CLUES AND QUESTIONS

DECIMALS

1. Changing the number 6.157 to 6.16 is an example of _____ .
2. In the number 5.17, the seven is in the _____ position.

FRACTIONS

1. The top number of a fraction is the _____ .
2. 12¼ is called a _____ .

GEOMETRY

1. A triangle has three sides, three angles, and three _____ .
2. A _____ is a chord that passes through the center of a circle.

A culminating activity that can bring closure to the topic is the reaction side of the anticipation/reaction guide. The teacher again reads the guide statements while students mark the sentences with which they agree. Class discussion can follow and any remaining misconceptions can be clarified. At this point the students can assess whether their concepts of the topic have changed and perhaps be able to ascertain why these changes have occurred.

Conclusion

Using pre-instruction and post-instruction strategies in the developmental mathematics classroom creates a learning atmosphere that increases teacher-pupil interaction while developing reading and mathematical skills. These techniques insure active student participation, the activation of prior knowledge, briding the gap between old and new information, and increased comprehension of the subject matter.

REFERENCES

Ausubel, P. (1960). The use of advanced organizers in the learning and retention of meaningful verbal material. *Journal of Educational Psychology*, 51, 267-272.

Bean, T.W. (1978). Developing content area vocabulary. *Journal of Reading*, 22, 102-103.

Bean, T.W. and Peterson, J. (1981). Reasoning guides: Fostering reading in content areas. *Reading Horizons*, 21, 196-199.

Earle, R.A., and Barron, R.F. (1973). An approach to teaching vocabulary in content areas. *Research in Reading in the Content Areas: Second Year Report*. Syracuse, New York: Reading and Language Arts Center, Syracuse University.

Moore, D.W., Readence, J.E., and Rickleman, R.J. (1982). *Pre-reading activities for content area reading and learning*. Newark, Delaware: International Reading Association.

Wade, S. (1983). A synthesis of the research for improving reading in social studies. *Review of Educational Research*, 53, 461-498.

Wood, K. and Mateja, J. (1983). Adapting secondary level strategies for use in elementary classrooms. *The Reading Teacher*, 36, 492-496.

MT 3. We all know what logarithms are and that they were developed by Napier in 1614. What was the original name for logarithms?

Reflections on the Studies of Comparisons of U.S. Students and Students in Other Countries

by James Baldwin
Nassau Community College, Garden City, NY 11530

Editor's Note: Although the studies cited in this article deal with students in grades K-12, the author and the editor feel that their research implications for mathematics education may be applied equally as well in the wide range of courses taught in two-year colleges.

Jim received his Ph.D. in Educational Research from Hofstra University in 1977. In addition to his regular teaching duties at Nassau Community College, he is greatly involved with in-service programs for teachers in the Long Island area. Jim is a former president of AMATYC.

During this past year a number of studies concerned with comparisons of students in the United States and those in other countries were completed and published. Some of these studies were discussed in January 1987, at a symposium called by the Mathematics Science Education Board. The purpose of this short article is to give an overview of three of these studies and their implications for mathematics educators. For a more complete discussion of each study please refer to the original study listed in the bibliography.

School Achievement and Socialization in Japan: Implications for Educators
by Kida, Shwalb, and Shwalb

In 1981, a total of 1700 Japanese students and 9582 Illinois students were administered the High School Mathematics Test developed by the Educational Testing Service. The 60 item test consisted of questions drawn from algebra, geometry, modern mathematics, data interpretation, and probability. In addition, a background questionnaire administered at the time of the test assessed the level of mathematics completed, frequency of leisure reading, reading self evaluation, frequency of discussions of schooling with parents, and perceptions of test comfort and importance. The data were analyzed by computing appropriate means, standard deviations, and correlations. Achievement scores were also regressed on all the other explanatory variables. The important results of the study as seen by the authors were:

1. Quantity of instruction and motivational variables were most important in determining mathematics achievement in both Japan and USA.
2. Older students did better than younger ones, and males outscored females in Japan.
3. At three age levels, the Japanese students had a two-standard-deviation achievement advantage over their USA counterparts.

Mathematics Achievement of Chinese, Japanese, and American Children
by Stevenson, Lee, and Stigler

This study was interested in students enrolled in kindergarten, first and fifth grades. One city was selected from each of the three countries: Minneapolis of the USA, Sendai of Japan, and Taipei of China. Ten schools were selected from each city and within each school two first grade and two fifth grade classes were randomly chosen. From each class

six boys and six girls were randomly selected yielding a total of 240 first graders and 240 fifth graders from each city. Elementary school is mandatory in each country and the entrance age to school is the same. Even though kindergarten is not mandatory in Japan and China, 98% of the Sendai 5-year olds and 80% of the Taipei 5-year olds attend kindergarten. Hence, 288 students in kindergarten were chosen from each city. To measure the achievement of the students in the samples, mathematics tests were developed for the kindergarten students and those in grades one and five. The kindergarten test contained items assessing basic concepts and operations included in the curriculum from K-3. Tests for grades one and five students contained 70 items derived from concepts and skills appearing in mathematics curriculum through grade 6. Each test was administered to one child at a time and was not timed. The student continued with the test to the point where over 1/4 of the items at the grade level were missed. In addition to the mathematics test being administered, observations were made of the classes during school, and interviews with the parents, students and techers were completed. The data derived from the study were analyzed through a series of means, standard deviations, percentages, and graphs. The important results of the study as seen by the authors were:

1. USA students are slightly behind their counterparts in kindergarten, but by the fifth grade they are surpassed by both the Japanese and Chinese students.
2. The ability of the students in all the countries are similar, but large differences exist in their school lives, the attitudes and beliefs of their mothers and teachers, and the involvement of both parents and children in schoolwork.

Mathematics Achievement in US Schools: Preliminary Findings from the Second International Mathematics Study
by Travers and McKnight

Two different population of students were surveyed in 22 different countries. The first population consisted of the students in the grade level at which a majority of them were 13 at mid year. The second population consisted of students in the secondary schools who were studying mathematics at least 5 hours per week. For the USA, this meant that the two populations were eighth graders, and twelfth graders who were enrolled in mathematics and had completed two years of algebra and one year of geometry. A sample of schools were randomly selected from across our country and approximately 7000 eighth grade and 5000 twelfth grade students were chosen. Three concerns of the study were the intended, the implemented, and the attained curriculums. To determine the intended curriculum, the curriculum guides, course outlines, textbooks and state mandated curriculums were analyzed. To determine the implemented curriculum, some questionnaires were distributed to teachers on topics studied, method of presenting specific mathematics topics, and the use of calculators. To determine the attained curriculum, achievement tests were designed. Also, a number of opinionnaires and attitude scales were devised to elicit students' views on the nature, importance, ease and appeal of mathematics in general. The data were studied by calculating appropriate means, standard deviations, percents and graphs. Some of the important results of the study as seen by the authors were:

1. USA students were slightly above the international average in computational arithmetic (calculation) and well below the international average in non-computational arithmetic (problem solving).
2. USA students achievement in algebra was comparable to the international average.

3. Achievement in geometry for the USA was among the bottom 25% of all countries, reflecting minimal coverage of this subject area.
4. USA students enrolled in calculus classes achieved at or near the average achievement of other advanced students in other countries. However, the USA students enrolled in precalculus classes (the majority of twelfth grade mathematics students), achieved substantially below the international average.
5. Overall attitude of USA students toward mathematics was positive.
6. In comparison to most other countries, the USA eighth grade curriculum is not unique. Four different mathematics programs each with very different mathematical content were identified.
7. The eighth grade mathematics curriculum in the USA tends to be "arithmetic-driven," resembling much more the end of elementary school than the beginning of high school.

These three studies are representative of the studies being completed and published on comparisons of mathematics students in various countries. In these and other studies that were discussed at the Symposium in Washington, various implications for mathematics education were given:

1. As educators we should stress the importance of the effort students put into their work rather than the intelligence of the students.
2. It is important to assign homework in mathematics at all levels of instruction and to make sure the students complete the homework.
3. Students should continue mathematics study as far as possible in order to keep doors of opportunity open.
4. In line with the previous implication, we should reconsider the widely diverse curriculums in the eighth grade that are permitting our students to select out of mathematics before they even reach our high schools and colleges.
5. There should be more concrete and intuitive geometry courses, so our students have a firmer foundation in geometry.
6. Cookbook rules to be memorized and used to solve problems must be reduced.
7. Elementary schools are the feeders of secondary schools and colleges. We must be concerned as teachers and citizens with our elementary school mathematics curriculum.

   a. The number of times a student is permitted out of an elementary school class for all sorts of other activities should be reduced.

   b. The amount of time that is devoted to giving directions or passing out materials should not count towards the mathematics instruction time.

   c. More concrete work with mathematics is necessary in early grades in order to build future abstractions on a firm foundation.
8. At all levels we should work to reduce the administrative tasks that take teachers away from their primary responsibility — teaching.

REFERENCES

Kida, H., Shwalb, D., Shwalb, B., (1985), "School Achievement and Socialization in Japan: Implications for Educators", Evaluation in Education: An International Review Series, Vol. 9, No. 3, Pergamon Press

Stevenson, H., Lee, S., Stigler, J., (1986), "Mathematics Achievement of Chinese, Japanese, and American Children," *Science*, Vol. 231, p. 693-698

Travers, K., McKnight, C., (1985), "Mathematics Achievement in US Schools: Preliminary Findings From The Second IEA Mathematics Study", Phi Delta Kappan, p. 407-413

McKnight, C., et. al., *The Underachieving Curriculum: Assessing U.S. School Mathematics From An International Perspective*, Stipes Publ. Co., Champaign, Ill. 61820

AUFMANN/BARKER HARDCOVER TEXTS:

THE MASTERY IS IN THE METHOD

Review courses featuring an instructional delivery system that results in higher levels of mastery, with less demand on the instructor's time.

Richard N. Aufmann and **Vernon C. Barker**
Both of Palomar College

Beginning Algebra with Applications
350 pages • hardcover • Study Guide • Instructor's Annotated Edition • Alternate Testing Program Solutions Manual • Computerized Test Generator GPA: Grade Performance Analyzer • 1986
Paperback version is available.

Intermediate Algebra with Applications
452 pages • hardcover • Ancillaries as above • 1986
Paperback version is available.

For adoption consideration, request examination packages from your regional Houghton Mifflin office.

Houghton Mifflin
13400 Midway Rd., Dallas, TX 75244-5165
1900 S. Batavia Ave., Geneva, IL 60134
925 E. Meadow Dr., Palo Alto, CA 94303
University Sq., 101 Campus Dr., Princeton, NJ 08540

REGULAR FEATURES

Timeless and Timely Issues in the Teaching of Calculus

by Gregory D. Foley
Ohio State University
Columbus, OH 43210

Preparation of this article was supported in part by grants from the National Science Foundation, the Ohio Board of Regents, The Ohio State University, and Standard Oil Corporation (Ohio). The opinions and conclusions herein expressed are those of the author and do not necessarily reflect the views of these funding agencies.

I thank the colleagues and associates who offered suggestions during the preparation of this article.

Yet another article on revising and revitalizing calculus education in the United States may seem redundant. So much has been written lately on the subject (see, e.g., Douglas, 1986; Page, 1984, 1985; Steen, 1987). My aim is not to repeat, but rather to provide some information germane to the current dialogue that might otherwise go unnoticed and to underscore some points already made by others. Calculus education faces both long-standing problems and new challenges. This article explores these timeless and timely issues as well as some possible solutions.

Timeless Issues

Many of the complaints we hear about the state of calculus education in the United States today are far from new. Woods's 1929 article "How Can Interest in Calculus Be Increased?" (republished in 1983) spoke of poorly prepared, "lethargic" students. The preparation problem has worsened due to the virtual disappearance of analytic geometry and the connections between algebra and geometry that it emphasized. Allendoerfer (1963) wrote of this disappearance, pointing out that in the first half of this century calculus was a course for college juniors, or sophomores who were able to complete college algebra, trigonometry, and analytic geometry in their freshman year. Some schools and colleges still offer a course in analytic geometry, but many do not; and what is worse is that most students are entirely missing the connections linking equations and their graphs. Achievement data from the Second International Mathematics Study (Chang & Ruzicka, 1985) show little knowledge or growth among United States precalculus 12th graders on the topics of coordinate geometry, functions, and graphing—prerequisites for success in calculus.

Allendoerfer (1963), Osgood (1907), Rietz (1919), and Woods (1929/1983) stressed the importance of student understanding and higher-order thinking as goals for calculus

itself. They expected students to gain a deep knowledge of calculus and to retain it, but knew this was often not the case. Today we have research results that confirm that students do not learn and retain calculus in the manner we would wish: Orton (1983a, 1983b) found misconceptions to be widespread among the 110 students he interviewed.

For a calculus course to be successful, it must be well taught. Perhaps, "mathematics, due to its inherent nature, needs a teacher more than any other subject" (Woods, 1929, p. 28). Grabiner (1983), however, made the provocative observation that mathematicians as a rule did not teach until the late 1700s when economic circumstances forced them into the classroom (p. 189). How many of today's mathematicians are teachers by choice and how many out of necessity?

You may have concluded thus far either that there is no current crisis in calculus or that there has always been a crisis, which we have just now begun to notice. Although many of the problems we face in teachng calculus have been around ever since we began to teach it, there are also some new issues in the current calculus debate.

Timely Issues

The shrinking of the computer, the emergence of discrete mathematics, and the advent of nonstandard analysis are all relatively new forces acting on calculus curriculum and instruction. These forces have given rise to new issues in the teaching of calculus. How can we meaningfully and effectively incorporate relevant new mathematics into the calculus course? How will microcomputers and hand-held computers change the face of calculus? Will these devices free us from computational drudgery and allow us to focus on critical concepts?

Microcomputers and hand-held computers can now not only "crunch" numbers but also manipulate symbols and sketch graphs. Moreover, these programs, known as computer algebra systems, are becoming increasingly easier to use. Small, Hosack, and Lane (1986) raised many questions about the possible implications of computer algebra systems for undergraduate teaching. What are the answers to these questions as regards calculus?

Wittman (1980) observed that "discrete functions arise empirically from measurement protocols which are contained, for example, in textbooks for science. Discrete functions are also obtained by *discretising* functions of continuous domains" (p. 390). So, if we are to show our students the connections between the calculus curriculum and the science curriculum, discrete analysis is called for. If we use numerical methods and computer graphics, we are discretizing calculus regardless of whether we tell our students. How much discrete analysis (difference and sum operators, etc.) should be included in elementary calculus?

Finally, where should the modern infinitesimal, or nonstandard analysis, approach to calculus fit into the emerging scheme of calculus instruction? This approach treats infinitely small and infinitely large quantities as bonafide numbers and exploits the resulting expanded number system in the study of calculus. (For elaboration of this issue, see Foley, 1986; Harnik, 1986; Schwarzenberger, 1978.) The current trend seems to be toward a numerical and graphical approach to calculus. Are these approaches compatible?

Possible Solutions

Preparation for Calculus

A timeless problem in calculus teaching has been inadequate student preparation. In order to solve this problem, we must first make room in the curriculum for the building of

appropriate student understandings including the interrelationships between algebra and geometry. After carefully analyzing three textbooks series, Flanders (1987) concluded that there is a disparity in the amount of new mathematics presented across Grades K through 9. Specifically, less than half of the mathematical content in Grade 2 and in Grades 4 through 8 is new material. In contrast, 90% of ninth-grade algebra content is new. Usiskin (1987) concluded that, with a thoughtful revision of the K-7 mathematics curriculum, the average student could study first-year algebra in Grade 8. Students could then complete 4 years of high school mathematics beyond beginning algebra.

In its October 1, 1987 draft, the Commission on Standards for School Mathematics of the National Council of Teachers of Mathematics called for a massive reform in the K-12 mathematics curriculum, which is consonant with Usiskin's observations. In particular, the Commission called for high school students to engage in an "informal exploration of calculus concepts from both a graphical and numerical perspective" (p. 128). This would occur as a part of a study of elementary functions and plane and solid analytic geometry. The author is currently involved in a curriculum project aimed at developing an interactive graphics-based precalculus course of this type — the Ohio State University Calculator and Computer Precalculus (C^2PC) Project. The C^2PC materials (Demana & Waits, 1987a) introduce students to such topics as extrema, monotonicity, asymptotic behavior, continuity, and limits (see Demana & Waits, 1987b, 1987c; Foley, in press; Waits & Demana, 1987).

Perhaps a "brave new world" with college freshmen who are prepared for the study of calculus is on the horizon. In the meantime, colleges, especially two-year colleges, will have an important role in bridging the gap between high school and university mathematics. Moreover, colleges (again, especially two-year colleges) will continue to serve students who have been taught via today's school mathematics curriculum well into the next century. So, we must design modern college courses to prepare our students for calculus.

Quality Teaching

Quality teaching requires a great deal of time and effort; such teaching must be rewarded, even by institutions with research as their primary mission. Rodi (1986) stated in his Tulane conference paper, "Nothing we discuss at this conference will have any impact on calculus instruction unless there is . . . a dramatic change in faculty involvement in undergraduate instruction" (p. 118). Faculties of most two-year colleges and many four-year colleges and universities do give much attention to calculus instruction. This needs to be the rule rather than the exception.

Testing and Grading

We must test what we wish our students to learn. Tests drive the curriculum. Much of the recent discussion concerning testing has focused on standardized tests, but we need to take a hard look at our own tests, too. Tests give students a "blueprint" of the actual objectives for the course (Shields, 1965). If we wish students to understand concepts, we must test for the understanding of concepts. And we need not limit ourselves to traditional testing and grading schemes. Reading, writing, talking, and listening are all needed to promote higher-order thinking, as are challenging problems that require a sustained effort. Lovelace and McKnight (1980) investigated the effects of reading instruction on calculus problem solving. Mett (1987) discussed some student writing activities aimed at deepening calculus understanding.

Curricular Alternatives

The current calculus curriculum is a curriculum of compromise. Each calculus textbook tries to please all consumers and ends up pleasing few. Now the consensus seems

to be that calculus textbooks must change (see Zorn, 1986). Perhaps the new forces acting on the calculus curriculum, together with the financial support of the Sloan Foundation through its sponsoring of conferences (Douglas, 1986; Ralston & Young, 1983; Steen, 1987) and the National Science Foundation through its calculus curriculum development program (1987), will lead to a variety of new calculus books that will serve our diverse needs.

Various alternatives have already been investigated. Some of these investigations have involved formal research, and some have resulted in the development of instructional materials. A bibliography of works not cited in this paper is provided to assist readers who wish to study curricular alternatives further.

Summary and Conclusions

We have seen that both timeless and timely issues must be addressed if we are to improve the teaching of calculus. We must rededicate ourselves to quality teaching and set high expectations for our students. The power of computer algebra systems offers new hope that we can focus the calculus course on comprehension and higher-order thinking. The precalculus curriculum, too, can be reshaped with a mix of something old—analytic geometry—and something new—computer graphics. Perhaps we will finally begin to encounter adequately prepared students in our calculus courses. Yet creating a modern calculus course that integrates classical, discrete, and nonstandard methods with computer graphics and symbolic manipulation programs will be no mean feat. We must blend the best of our collective wisdom using relevant information gleaned from educational research and cognitive psychology. For now we can continue to test various approaches and share our successes and our concerns.

References

Allendoerfer, C. B. (1963). The case against calculus. *Mathematics Teacher, 56,* 482-485.

Chang, L.-C., & Ruzicka, J. (1985). *Second international mathematics study: United States technical report I.* Champaign, IL: Stipes Publishing.

Commission on Standards for School Mathematics. (1987). *Curriculum and evaluation standards for school mathematics* (working draft). Reston, VA: National COuncil of Teachers of Mathematics. (Available from Thomas A. Romberg, Chair, NCTM Commission on Standards for School Mathematics, 1906 Association Drive, Reston, VA 22091)

Demana, F., & Waits, B. K. (1987a). *Advanced high school mathematics* (pilot version). Unpublished manuscript, The Ohio State University, Department of Mathematics, Columbus.

Demana, F., & Waits, B. K. (1987b). Enhancing problem solving skills in mathematics through microcomputers. *Collegiate Microcomputer, 5,* 72-75.

Demana, F., & Waits, B. K. (1987c). Problem solving using microcomputers. *College Mathematics Journal, 18,* 236-241.

Douglas, R. G. (Ed.). (1986). *Toward a lean and lively calculus: Report of the conference/workshop to develop curriculum and teaching methods for calculus at the college level* [MAA Notes No. 6]. Washington, DC: Mathematical Association of America.

Flanders, J. R. (1987). How much of the content in mathematics textbooks is new? *Arithmetic Teacher,* 35(1), 18-23.

Foley, G. D. (1986) [Review of *Elementary Calculus: An Infinitesimal Approach* (2nd ed.] *The AMATYC Review,* 8(1), 60-62.

Foley, G. D. (in press). Educational reflections—Future shock strikes again: Hand-held computers. *The AMATYC Review.*

Grabiner, J. V. (1983). Who gave you the epsilons? Cauchy and the origins of the rigorous calculus. *American Mathematical Monthly, 90,* 185-194.

Harnik, V. (1986). Infinitesimals from Leibniz to Robinson: Time to bring them back to school. *The Mathematical Intelligencer, 8,* 41-47, 63.

Lovelace, T. L., & McKnight, C. K. (1980). The effects of reading instruction on calculus students' problem solving. *Journal of Reading, 23,* 305-308.

Mett, C. L. (1987). Writing as a device in calculus. *Mathematics Teacher, 80,* 534-537.

National Science Foundation. (1987). *Program announcement—Undergraduate curriculum development in mathematics: Calculus.* Washington, DC: Author, Division of Mathematical Sciences. (Available from Forms and Publications Unit, NSF, 1800 G Street, NW, Washington, DC 20550)

Orton, A. (1983a). Students' understanding of integration. *Educational Studies in Mathematics, 14,* 1-18.

Orton, A. (1983b). Students' understanding of differentiation. *Educational Studies in Mathematics, 14,* 235-250.

Osgood, W. F. (1907). The calculus in our colleges and technical schools. *Bulletin of the American Mathematical Society, 13,* 449-467.

Page, W. (Ed.). (1984). Forum. *College Mathematics Journal, 15,* 370-399.

Page, W. (Ed.). (1985). Forum. *College Mathematics Journal, 16,* 1-21.

Ralston, A., & Young, G.S. (Eds.). (1983). *The future of college mathematics: Proceedings of a conference/workshop on the first two years of college mathematics.* New York: Springer-Verlag.

Rietz, H. L. (1919). On the teaching of the first course in calculus. *American Mathematical Monthly, 19,* 341-344.

Rodi, S. (1986). Some systemic weaknesses and the place of intuition and applications in calculus instruction. In R. G. Douglas (Ed.), *Toward a lean and lively calculus: Report of the conference/workshop to develop curriculum and teaching methods for calculus at the college level* [MAA Notes No. 6] (pp. 115-127). Washington, DC: Mathematical Association of America.

Schwarzenberger, R. L. E. (1978). The impact of non-standard analysis on school calculus. *Mathematics Education for Teaching, 3*(2), 3-12.

Shields, M. R. (1965). *The construction and use of teacher-made tests.* New York: National League of Nursing.

Small, D., Hosack, J., & Lane, K. (1986). Computer algebra systems in undergraduate instruction. *College Mathematics Journal, 17,* 423-433.

Steen, L. A. (Ed.). (1987). *Calculus for a new century: Background papers for a national colloquium, October 28-29, 1987.* Unpublished manuscript, Board on Mathematical Sciences and the Mathematical Sciences Education Board of the National Research Council. (Revisions of these papers to appear as part of *Calculus for a new century* to be published by the Mathematical Association of America)

Usiskin, Z. (1987). Why elementary algebra can, should, and must be an eighth-grade course for average students. *Mathematics Teacher, 80,* 428-438.

Waits, B. K., & Demana, F. (1987). Editorial: Solving problems graphically using microcomputers. *UMAP Journal, 8*(1), 1-7.

Wittman, E. (1980). 'Discrete analysis'—An approach to fundamental ideas of analysis. *Educational Studies in Mathematics, 11,* 383-392.

Woods, R. (1929). How can interest in calculus be increased? *American Mathematical Monthly, 36,* 28-32. (Reprinted in *Mathematics in College,* Winter, 1983, 31-35.)

Zorn, P. (1986, March-April). Calculus redux. *Focus: The Newsletter of the Mathematical Association of America,* pp. 1-2, 5.

Bibliography

Committee on the Undergraduate Program in Mathematics Panel on Calculus Articulation. (1987). Report of the CUPM Panel on Calculus Articulation: Problems in the transition from high school calculus to college calculus. *American Mathematical Monthly, 94,* 776-785.

Edge, O. P., & Friedberg, S. H. (1984). Factors affecting achievement in the first course in calculus. *Journal of Experimental Education, 52,* 136-140.

Ferrini-Mundy, J. (1987). Spatial training for calculus students: Sex differences in achievement and in visualization ability. *Journal for Research in Mathematics Education, 18,* 126-140.

Frantz, M. E. (1986). Interactive graphics for multivariable calculus. *College Mathematics Journal, 17,* 172-181.

Freed, W. (1985). *Infinitesimal Calculus 1* (version 2.2). Unpublished manuscrpt, Concordia College, Department of Mathematics, Edmonton, Alberta.

Foley, G. D. (1987). Using infinitesimals to introduce limits and continuity in a community college calculus course (Doctoral dissertation, The University of Texas at Austin, 1986). *Dissertation Abstracts International, 47,* 3345A.

Hamadanizadeh, J. (1980). The systems approach to teaching calculus. *International Journal of Mathematical Education in Science and Technology, 11,* 69-73.

Henle, J. M. (1982). *The calculus of infinitesimals.* Unpublished manuscript, Smith College, Department of Mathematics, Northampton, MA.

Hirsch, C. R., Kapoor, S. F., & Laing, R. A. (1983). Homework assignments, mathematical ability, and achievement in calculus. *Mathematics and Computer Education, 17,* 51-57.

Keisler, H. J. (1986). *Elementary calculus: An infinitesimal approach* 2nd ed.). Boston, MA: PWS Publishers.

King, L. (1983). The epsilon-delta connection. *Two-Year College Mathematics Journal, 14,* 42-47.

Palmiter, J. R. (1986). The impact of a computer algebra system on college calculus. *Dissertation Abstracts International, 47,* 1640A.

Patterson, D. L. (1984). Calculus students' utilization of graphs as visual representations. *Dissertation Abstracts International, 44,* 3311A-3312A.

Ponte, J. P. M. da. (1984). Functional reasoning and the interpretation of Cartesian graphs. *Dissertation Abstracts International, 45,* 1675A.

Spresser, D. M. (1979). Placement of the first calculus course. *International Journal of Mathematical Education in Science and Technology, 10,* 593-600.

Stenger, W. (1980). The computation of limits in elementary calculus. *Mathematics Teacher, 73, 615-617.*

Sullivan, K. A. (1975). The teaching of elementary calculus: An approach using infinitesimals. *Dissertation Abstracts International, 35,* 6443A. (University Microfilms No. 75-2,484)

Sullivan, K. A. (1976). The teaching of elementary calculus using the nonstandard analysis approach. *American Mathematical Monthly, 83,* 370-375.

Tall, D. O. (1980). Looking at graphs through infinitesimal microscopes, windows and telescopes. *Mathematical Gazette, 64,* 22-49.

Tall, D. O. (1980). The notion of infinite measuring number and its relevance in the intuition of infinity. *Educational Studies in Mathematics, 11,* 271-284.

Tall, D. O. (1985, March). Understanding the calculus. *Mathematics Teaching,* No. 110, 49-53.

Tall, D. O. (1985, June). The gradient of a graph. *Mathematics Teaching,* No. 111, 48-52.

Tall, D.O. (1985, December). Tangents and the Leibniz notation. *Mathematics Teaching,* No. 113, 48-51.

Tall, D. O. (1986, March). A graphical approach to integration. *Mathematics Teaching,* No. 114, 48-51.

Tall, D. O. (1986, June). Lies, damn lies . . . and differential equations. *Mathematics Teaching,* No. 115, 54-57.

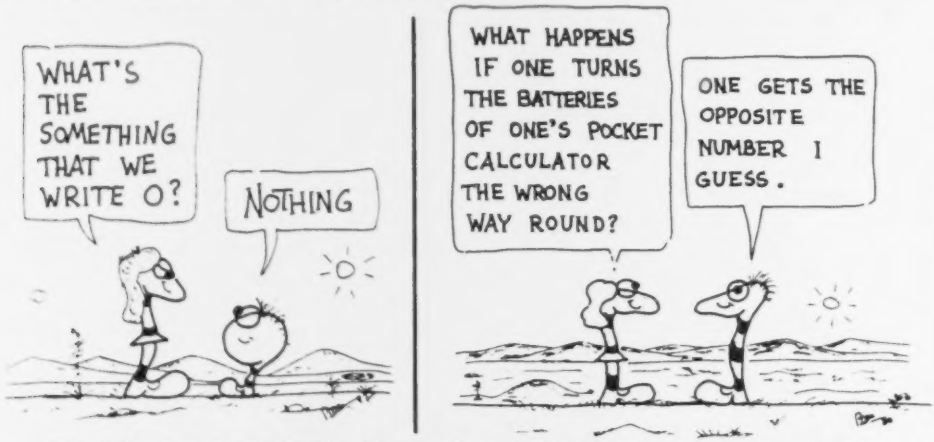
Tall, D. O. (1987, March). Whither calculus? *Mathematics Teaching,* No. 118, 50-54.

Pandora's Rectangular Parallelipiped

Editor: $\int_{\theta}^{NER} SEC^2x\ dx$

The fact that this is the second consecutive **PRP** column may well be due to that fact that it was submitted before the first one was published. In any event, publication of winning entries in the "New Metric Unit" contest proposed in the last **PRP** column (which was also the first column, making it the first and last column) will appear next time.

In the meantime, meet Andrejs Dunkels and the "Footies". Professor Dunkels teaches mathematics at the University of Lulea, in northern Sweden. Among his many varied interests — for example, he is a musician, speaks Swahili, and collects yo-yos — he is a cartoonist. He has been drawing cartoons of the "Footies" for over 15 years; many have appeared in publications throughout Europe. "Footies" cartoons will be a regular feature of this column, their first — but certainly not their last — exposure in print in the U.S. (Andrejs Dunkels is Latvian. As a consequence of Swedish friends addressing him by his initials, Andrejs' cartoons are signed "Adde" — A.D.)


WHAT'S THE SOMETHING THAT WE WRITE O?
NOTHING
WHAT HAPPENS IF ONE TURNS THE BATTERIES OF ONE'S POCKET CALCULATOR THE WRONG WAY ROUND?
ONE GETS THE OPPOSITE NUMBER I GUESS.

The following should inspire dozens of entries in this column's proposed "Mathematical Limerick" contest. Winning entries will be published in the second issue following this one.

The math teacher's oldest son, Hugh,
To demonstrate what he could do,
Wrote (500 x 7)
- (9 x 11)
= 3402

When the prof saw what Hugh had done,
He sighed and said: "Number one son,
You'll never pass
This algebra class —
Your answer is too big by
an amount equal to the multiplicative
identity for the set of real numbers!!

There is a column in a well-known outdoors magazine called "Exit Laughing". If this portion of **PRP** could have a title, it would certainly be "Exit Groaning", as you will soon see.

Two women from the Poo-Owl tribe (the night-shift branch of the Poo-Hawks) were sitting one day on two buffalo robes; nearby sat a third, who would nowadays be referred to as "queen-sized", resting on a large skin recently given her by a friend from the Congo. Observed the tribe's inept lichen-collector, Rolling Stone: "It appears to me that the squaw on the hippopotamus is equal to the sum of the squaws on the other two hides."

Contributions to this column, entries to the "Mathematical Limerick" contest, and other items equal in value to what you have seen so far, may be sent to:

Leslie R. Tanner, Editor
PRP Column, ***The AMATYC REVIEW***
c/o The College of Idaho
2112 Cleveland Blvd.
Caldwell, ID 83605

MT 4. Who invented the first digit using mechanical calculating devise? In which year?

Book Reviews

Edited by John J. Edgell, Jr.

USING LOTUS 1-2-3, WORDSTAR, AND dBASE III PLUS, by Thomas N. Warrner and D. Michael Werner. Scott, Foresman and Company, Glenview, Illinois, 1987. 258 pp.; ISBN 0-673-18761-6.

For those of us who are searching for a book which deals strictly with Lotus 1-2-3, WordStar, and dBase III Plus, as well as DOS, search no more. This book is a "no-frills" look at some very popular software packages. Your students will appreciate that, because they can immediately get down to work with the application. This is not a computer literacy book.

These are demonstration programs which accompany the book, all easy to obtain and there are plenty of appropriate examples provided within each package.

Lotus users will learn practically all there is to know about Lotus 1-2-3 except for Lotus' programmability. However, you will learn how to graph and sort, two features often overlooked in books of this nature. The coverage of WordStar is adequate. The one problem I can see is the authors should have chosen WordPerfect as the word processor. WordPerfect is now the number one selling word processor, because it is easier to learn and use when compared to WordStar.

dBase III Plus, often confusing to students, is studied after the Lotus 1-2-3 database management system is presented. This simplifies the discussion of dBase commands, and lends itself well to the comparison of the Lotus database environment and the dBase III Plus package. Both sorting and indexing are covered.

The objection to this book is it is sometimes confusing to determine if the authors want the reader to enter a command, or just read the command but not type anything. Thus, at times, readers may type something when they should be reading.

In conclusion, go out and get the book and use it in your class. It's not 700 pages long as some computer information texts which contain tutorials. It's small, easy to carry, and good.

Reviewed by Professor Louis V. Gioia, Math Department, Nassau Community College, Garden City, New York 11530.

FORTRAN 77 FOR HUMANS, Third Edition, by Rex Page, Rich Didday, Elizabeth Alpert, West Publishing Company, St. Paul, Minnesota, 1986, 462 pp; ISBN 0-314-93404-9.

I am often disappointed with computer programming books which claim to be written for the "ordinary individual", or the "human". These descriptions are often given to books which are anything but understandable and easy to learn from. This book is one exception. Its salient quality is that it is truly easy to read and understand. This fact, coupled with the "soft" print style, are my reasons for recommending **FORTRAN 77 FOR HUMANS** as a primary text for one semester Fortran 77 courses.

The organization of the chapters is most logical. I had little objection to their order. When reading the book, one will find a general introduction to Fortran 77 by way of example, followed by the study of data types, loops, formatting, arrays, subprograms, and files. There are other selected topics one might be interested in such as the debugging of Fortran programs and arrays as data structures.

Of particular interest to me was the fine treatment of subprograms. I found the presentation to be most suited to my students' needs. The examples were well chosen, and not too difficult to follow.

My one major objection to the book was its lack of numerous short answer exercises. There are numerous program examples covering a wide variety of applications, but not enough of the small, fact type short answer questions one often finds in a programming textbook.

In all, the Fortran 77 instructor looking for a "can't miss" Fortran 77 book would do very well to look at, and choose this one.

Reviewed by Professor Louis V. Gioia, Nassau Community College, Garden city, New York 11530.

APPLIED FINITE MATHEMATICS, by George J. Kertz. West Publishing Co., St. Paul, Minnesota, 1985. 530 pp.; ISBN 0-314-85255-7.

This book is written for students of management, biological, and social sciences who have background in algebra. Calculus background is not required. Chapter 1 is a fun chapter on graph theory and applications of mathematical models. The main purpose of this chapter is to illustrate the author's belief that the students should be taught to express themselves mathematically by developing insight into the applications of mathematics in solving "real" life problems. Chapters 2-4 are on linear programming; chapters 5-7 are on probability and statistics; chapter 8 deals with mathematics of finance; chapter 9 is on logic; and chapter 10 is on functions and graphing. An algebra review is provided in one appendix and many numerical tables are provided in another appendix. All topics are carefully developed with plenty of depth. There are three types of exercises: (1) drill, (2) applications, and (3) discussion. Answers for all odd-numbered exercises are given. There is an ample supply of exercises.

The book may be used as a text for two independent courses where the topics may be selected depending on the instructor's taste.

An important feature of this book is that an emphasis is placed on the teaching of mathematical expression and verbalization of problems occurring in other disciplines. There are plenty of verbal problems and examples to emphasize this point of view. Other interesting features are: (1) treatment of Markov chains, (2) game theory, and (3) hypothesis testing about the mean.

The text is certainly well written, and I got the distinct impression that the author has enjoyed its writing, since it expresses his pedagogical philosophy mentioned above for his approach to teaching this material. Manifestly, I sensed a strong sense of enthusiastic motivation in the presentation of material and make-up of examples and exercises. The book will well serve its students, especially those who share in this excellent approach to mathematics.

Reviewed by Professor S. Singh, Department of Mathematics, Southwest Texas State University, San Marcos, Texas 78666.

ELEMENTARY LINEAR ALGEBRA, by Richard O. Hill, Jr., Academic Press, Inc., New York, New York, 1986. 404 pp.; ISBN 0-123-48460-X.

This textbook on elementary linear algebra covers most of the topics traditionally covered in a first course in the subject. The book is divided into six chapters and there are four appendices. Here is a brief overview of the material covered and its organization. Chapter 1 treats linear systems, matrices, Gaussian elimination, inverses of matrices, Gaussian elimination as a matrix factorization, properties of matrices and numerical considerations related to computer programming. Chapter 2 covers basic topics on vector spaces such as linear independence, span, subspaces, null spaces, basis, dimension, rank of a matrix, and an application to error-correcting codes. Chapter 3 deals with matrices as linear transformations, inner products, projections, orthogonal bases and matrices, Gram-Schmidt process, and a geometric approach to encoding the QR decomposition. Chapter 4 treats eigenvalues and eigenvectors after a brief introduction to determinants, diagonali-

zation and symmetric matrices. Applications to difference equations, differential equations, and quadratic forms are also given. Moreover, a numerical approach to the eigenvalue problem is presented. Chapter 5 deals with determinants. Chapter 6 covers more abstract topics such as function spaces, singular value decomposition, generalized inverses, vector spaces and linear transformations over an arbitrary field. A brief discussion of software and codes for linear algebra is given in an appendix.

The book is designed for students who have finished a calculus course, preferably the entire calculus sequence. Perhaps, the most important prerequisite for taking this course from this text is mathematical maturity. It is contended that the majority of students taking elemenary linear algebra are "technically oriented majoring in engineering or in the physical, biological, or mathematical sciences or in economics."

The intent of the text is to provide ample theoretical and applied background for these students which will suffice for later applications of linear algebra to their areas of study. Theoretical discussions and proofs are given. More complicated proofs are postponed until the end of the sections.

There are plenty of exercises and examples. The list of topics will give the instructor enough flexibility to tailor the course according to the needs of the students. A useful feature of the book is that all definitions, theorems, and other important facts are highlighted. There is no doubt that the text will prove useful to its intended audience.

Reviewed by Professor S. Singh, Department of Mathematics, Southwest Texas State University, San Marcos, Texas 78666.

Send reviews to:

Dr. John J. Edgell, Jr.
Editor, Book Reviews
The AMATYC Review
Mathematics Department
Southwest Texas State University
San Marcos, Texas 78666

MT 5. Who said, "Logic is logic, that's all I say."?

The Problem Section

Edited by Michael W. Ecker, Ph.D.
Mathematics Department
Pennsylvania State University
Wilkes-Barre Campus
Lehman, PA 18627

Welcome to another Problem Section! We have lots of goodies in store for you this issue!

First, thanks to those of you sending in problem proposals and solutions. I still could use even more problem proposals, but I am pleased that, six years after I founded this problem department, things may be perking up. Part of this, I suspect, is attributable to the rise in awareness of AMATYC's presence by and in the mathematics community. I'd like to think that it has something to do with increased pleasure with the efforts of the good readers of this section.

In any case, problem-solving activity **has** begun to pick up in this section, with a rise in set L solutions this past year. By the time you read this, set M will have been out for a while. It will be interesting to see whether you're all getting more involved, or whether set L was a statistical blip — a quirk.

This issue I have more proposals than usual, and I think you'll like the improved selection in terms of quantity and quality. There are more problems, several of them at a more reasonable level. For those of you computerizing, I welcome computer-related submissions, but bear in mind that computer activity here should complement or supplement, not supplant, the mathematics.

As the owner of nine computers, I enjoy both straight math and computing, of course. But if you're more into computer recreations than problem solving, contact me and I'll send you free, a complimentary issue of the *Recreational & Educational Computing Newsletter* (published 8 times per year, $24, with samples ordinarily sold for $3.50). Please supply a self-addressed, stamped envelope with 39 cents worth of postage affixed when you write. Please identify yourself as an AMATYC member or reader of this problem section.

Okay, on with the customary business of a problem section. I remind you that the *AMATYC Review* Problem Section seeks lively and interesting problems and their solutions from all areas of mathematics, but particularly those which have applicability of some kind in the lives of two-year college math faculty and students. Please eschew routine and familiar problems, however. If you're not sure, send something in anyway. At the least, we may be able to modify it to turn a routine question into a real think piece; at best, you may have a true gem that will bring you instant fame among your peers.

Send your problems, solutions, comments, generalizations, special cases, observations, and improvements to:

Dr. Michael W. Ecker
Problem Section Editor
The AMATYC Review
129 Carol Drive
Clarks Summit, PA 18411

Though I'm still using my home address, I'm pleased to share the news that, one year after returning to Penn State, I have had tenure re-conferred upon me.

For those interested in a resource for the basics of problem solving, my latest book, *Getting Started in Problem Solving and Mathematics Contests,* published by Franklin Watts (New York), copyright 1987, should be out even before you read this. Price should be not much more than $10, plus a couple of dollars for shipping and handling if you buy it mail order. If any reader wants to pre-order a copy, send a check to me, made payable to

Dr. Michael Ecker, for $13. The hardcover book includes problem-solving techniques and examples, pattern recognition, working backwards, using lists appropriately, and much more, with examples chosen from conventional math and even recreational math, including making change, the eccentric jailer, counting problems, a few teasers, etc. There are 128 pages, an index, and a beautiful jacket designed by the publisher.

Now let's bring on the problems . . .

Set N Problems. Due for ordinary consideration in summer of 1988, solutions to these problems may be submitted as late as Feb. 1989, since I plan to play catchup in coming issues. Of course, no problem is ever closed permanently, and new insights are always welcome. Plus, if there ever is a production delay, we may extend the deadine appropriately.

Problem N-1. Proposed by the Problem Editor, Pennsylvania State University, Wilkes-Barre Campus.

Consider the experiment of picking a positive integer. Suppose that the probability P(n) of selecting any positive integer n>1 satisfies P(n) = .99xP(n-1). a) What must be P(1), the probability of picking 1? b) In picking a positive integer, what is the expected value?

Problem N-2. Proposed by the Problem Editor.

According to a psychological experiment announced in the last couple of years, it was found that the average person perceives a rectangular room as "bigger" than a square room of the same area. One theory offered was that the average distance between two points in the rectangular room is supposed to be larger than for the square. Confirm or disprove this by calculating the expected length for both.

Caution: In many such probability and expected value questions, the key is to have an appropriate definition and setup for the expected length.

Problem N-3. Proposed by Stanley Rabinowitz, Alliant Computer Systems, Littleton, MA.

Over C, the field of complex numbers, the polynomial $x^2 + y^2$ factors as $(x+iy)(x-iy)$. Does the polynomial $x^2 + y^2 + z^2$ factor over C?

Hint: This is not as hard or theoretical as it may sound; think of undetermined coefficients. Give it a try!

Problem N-4. Proposed by the Problem Editor and another reader (whose name did not appear on the proposal) — edited.

$$\begin{pmatrix} 1 & 2 & 3 & \dots & n \\ n+1 & n+2 & \dots & & 2n \\ \dots & \dots & \dots & & \\ \dots & & & & \\ \dots & \dots & \dots & \dots & n^2 \end{pmatrix} = A$$

Consider the first n^2 positive integers, and create a square matrix A, above.

What is the value of det A, the determinant of A? Can you generalize?

Problem N-5. Proposed by Susan R. Ecker, Clarks Summit, PA.

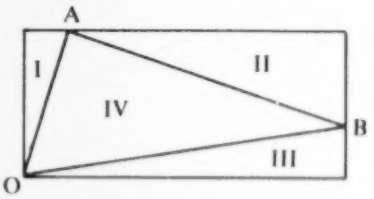
I observed that my daughter Melody, my son David, husband Mike (the problem editor guy), and I were born in December, January, February, and March, respectively. Is

this special or unusual? That is, if we make the plausible assumption that one is equally likely to be born in any month as any other (even if not exactly so), calculate the probability of a family of four, such as ours, having birthdays in four consecutive months.

Problem Editor's Comment: My wife comes up with a good question now and then, and considering that she's almost a mathaphobe, it's all the more remarkable.

Problem N-6. Proposed by Robert Carman, Santa Barbara, CA.

Points A and B on the sides of the rectangle shown are chosen so that triangles I, II, and III have equal area. What is the ratio of the area of triangle I to the area of triangle IV?



Problem N-7. Proposed by Paul Goodman, Skyline College, San Bruno, CA.

Prove that $\cos(\sin x) > \sin(\cos x)$ for all real x.

Problem N-8. Proposed by Glen E. Mills, Orange Co. Public Schools, Orlando, FL.

Verify the identity: $\sin(8x)[\cos x + \cos(3x) + \cos(5x)] = \sin(6x)[\cos x + \cos(3x) + \cos(5x) + \cos(7x)]$

Problem N-9. Proposed by the Problem Editor.

In a tennis match, if A beats B when A is serving, A is said to hold serve, but if B is serving when A wins, A is said to break B.

Prove the intuitively obvious fact that, after an even number of games have been played, if A and B have both won the same number of games, then each has broken the serve of the other the same number of times (which may be zero).

Problem N-10. (Prisoner's Dilemma) Proposed by several people at different times, but recently sent in by Steve Plett of Fullerton College, Fullerton, CA, for William Gearhart and Jack Walker (student), California State University, Fullerton.

You are given $u \geq 2$ urns, $w \geq u$ white balls, and $b > 0$ black balls. A prisoner is told that he may distribute the balls among the urns as he wishes. When he has finished, somebody will randomly select an urn, and then, from that urn, randomly select a ball inside. If the ball is black, the prisoner will be executed. If it is white, he will be paroled. The only restriction is that each urn must contain at least one ball. How should the prisoner distribute the balls so as to maximize the probability of his own survival?

Comment: Before my problem editing days and back in my problem solving days, I solved this in another publication (I won't say which, but it was within the last decade). I had pointed out to the problem editor then that a proof for the strategy was required. Please don't forget to back up **your** proposed distribution!

Solutions

Problem I-3, the one from a couple of years ago dealing with congruence relations, never garnered any solutions. I am withdrawing it for use in another publication, but if I have a complete solution, or if you have one that you would like to compare, I will communicate with you privately.

Likewise, Problem J-2 fared no better. It asked for whether a palindrome must be a multiple of 11. Since 232 isn't, the answer is no, in general, but I believe that the result is true if the palindrome has an even number of digits.

Geometric Paper-Folding

Problem J-3. Borrowed by the Problem Editor.

Given a square piece of paper, fold it as few times as possible so as to introduce creases forming angles (with themselves and/or the sides of the paper) of $\mathrm{Tan}^{-1}\,1$, $\mathrm{Tan}^{-1}\,2$, and $\mathrm{Tan}^{-1}\,3$.

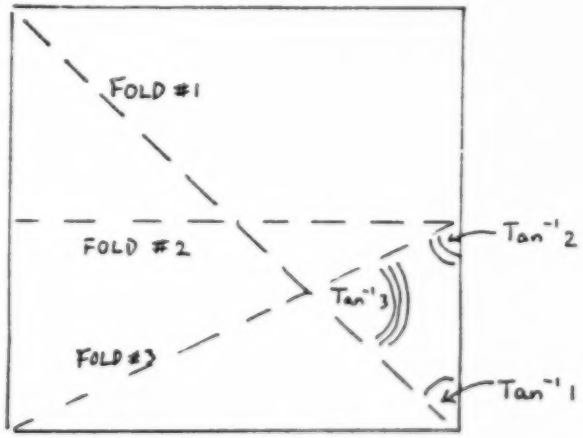
Solution by the Proposer (no others received), with a comment.

This problem appeared quite a few years ago in **Crux Mathematicorum**. The featured solver accomplished this task in four folds, as did I. However, I also accomplished this in three folds, but, for some reason, this superior solution was never used.

Everybody knows how to start: Form a diagonal to create an angle of $\mathrm{Tan}^{-1}\,1 = \pi/4$, a 45-degree angle. Next, since we need a 2:1 ratio of sides ("opposite over adjacent"), fold the paper in half to locate the midpoint of a side, and then fold from a corner to that midpoint to get $\mathrm{Tan}^{-1}\,2$ involved.

Oops! Seems as though I used up my claimed three folds, so how am I to get the third angle? The key to the solution is to note the interesting relation $\mathrm{Tan}^{-1}\,1 + \mathrm{Tan}^{-1}\,2 + \mathrm{Tan}^{-1}\,3 = \pi$(or, if you absolutely insist, 180 degrees). To prove this, it suffices to show that $\mathrm{Tan}^{-1}\,2 + \mathrm{Tan}^{-1}\,3 = 3\,\pi/4$. This, however, is easy to check using the fact that Tan (the principal tangent function, the restriction of the ordinary tangent; note the capital "T") is one-to-one, by construction. So, just use the formula for the tangent of a sum applied to the left side of the last equation to get -1, whose inverse tangent is indeed $3\pi/4$.

So, therefore, make sure that the first two angles are part of a triangle, and the third angle is then automatically $\mathrm{Tan}^{-1}\,3$. Here is the diagram.



Beautiful, isn't it?

Let Me Count the Ways . . .

Problem J-4. Borrowed by the Problem Editor.

Let N be the set of natural numbers. Is it true that, for each natural number n, there exists a polynomial $f_n(x_1,x_2,\ldots,x_n)$ in n variables such that the mapping $f_n: N^n \to N^n$ is a bijection? (Hint: Consider n=2 first.)

Solution by the proposer (no others received), with a comment.

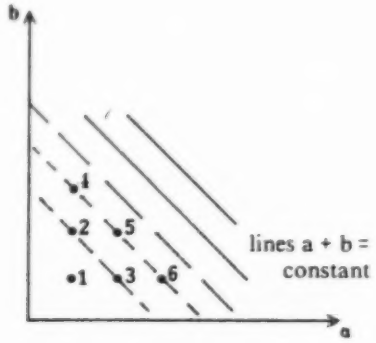
This one also was in **Crux,** but again I felt that the editor missed the boat with respect to publishing a useful solution. (If you're like me, you hate the ones that go: "This is a corollary of theorem XXX on page YYY of book ZZZ by Professor PPP at University UUU.")

As with many inductions, the crux of the matter is to be able to go from n=1 to n=2. The statement is certainly true, after all, for n=1. E.g. Just take f(x)=x. If we know how to go from 1 to 2 – i.e. if f_2 exists with the desired properties, then we will have finished, because we can then complete the induction step from n–1 to n by taking f_n to be defined by

$$f_n(x_1, x_2, \ldots, x_{n-1}, x_n) = f_2(f_{n-1}(x_1, x_2, \ldots, x_{n-1}), x_n).$$

(If f_{n-1} is a polynomial, then so is f_n. It is clear how the appropriate properties are inherited.)

To finish, take f_2 to be $f_2(a,b) = a + (\frac{1}{2})(a+b-2)(a+b-1)$.
To see how I found this — and why f_2 has the desired properties — consider the diagram. The object was to assign numbers 1, 2, and so on to the lattice points as indicated. To do so, we consider each line of the form a + b = constant (constant = 2, 3, ...). What we do is first give a value according to which such line a particular lattice point lies on, then add successively larger values to obtain distinct function values at the different points along each such line. To ensure that f_2 be bijective, we use the triangular numbers, T_k, with k=a+b–2, for the former part, and add the first coordinate for the latter part.



More Polynomial Nonnegativity

Problem J-5. Proposed by William P. Wardlaw, U.S. Naval Academy, Annapolis, MD.
Let f(x) be a polynomial of degree n such that $f(x) \geq 0$ for all real x. Prove that:
a. $f(x) + f'(x)/1! + f''(x)/2! + \ldots + f^{(n)}(x)/n! \geq 0$ for all real x, and
b. $f(x) + xf'(x)/1! + x^2f''(x)/2! + \ldots + x^nf^{(n)}/n! \geq 0$ for all real x.

Solution by proposer, with similar solution by David Price, Tarrant County Junior College, Ft. Worth, Tx.

The Taylor series expansion of f(u) around the point a is

$$f(a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!} (u-a)^k .$$

a. Letting u = x+1 and a = x, we see that

$f(x) + f'(x)/1! + f''(x)/2! + \ldots + f^{(n)}(x)/n! = f(x+1) \geq 0$ by hypothesis.

b. Letting $u = 2x$ and $a = x$, we see that

$f(x) + xf'(x)/1! + x^2f''(x)/2! + \ldots + x^nf^{(n)}(x) = f(2x) \geq 0$ by hypothesis.

Reference: AMATYC Problem G-4, **THE AMATYC REVIEW,** Vol. 6, No. 1, Fall/Winter 1984, p. 55.

Comment:

The identity used to obtain the solution for a was obtained in an abortive attempt at the solution to AMATYC Problem G-4. The rather pleasing identity used for part b was the second part of a solution to a problem posed on a recent Virginia Polytechnic Institute Mathematics Competition.

The Pizza-Boy's Tip

L-1. Proposed by Stephen Plett, Fullenton College, Ca.

Tony's giant-sized, thin-crust pizza is delivered in a 2' x 2' square-based box. When the circular pizza is set centered in the box, any extra pizza is folded over relative to the perimeter of the box. Such folded portions, including the portions underneath, are given to the delivery boy as a snack. What radius will maximize the portion which the customer retains?

Composite of essentially identical solutions by Glen E. Mills; Ann E. Watkins (Los Angeles Pierce College, Woodland Hills, CA); Robert A. Carman; John Brunn (Hayward College, CA); and the proposer.

To begin, $1 < r < 2$ and the area of the pizza which the customer retains is

$$A = \pi r^2 - 4r^2\left[2\,\mathrm{Cos}^{-1}\left(\frac{1}{r}\right) - \sin\left(2\,\mathrm{Cos}^{-1}\left(\frac{1}{r}\right)\right)\right],$$

whereas the area of the segment which the delivery boy gets as a snack is

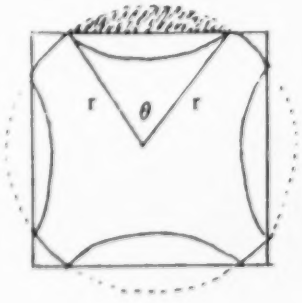
$8\left(\frac{1}{2}r^2\,(\theta - \sin\theta)\right), \theta = 2\,\mathrm{Cos}^{-1}\left(\frac{1}{r}\right)$.

Note that $\frac{1}{2}r^2(\theta - \sin\theta)$ is the area of a circular segment and there are 8 segments that the delivery boy gets as a snack.

$$\text{Now, } \frac{dA}{dr} = 2\pi r - \frac{8r}{\sqrt{r^2-1}} - 16r\,\mathrm{Cos}^{-1}\left(\frac{1}{r}\right) + \frac{16 - 8r^2}{r\sqrt{r^2-1}} + \frac{16\sqrt{r^2-1}}{r}$$

Equating dA/dr to 0, we get

$$r = \frac{1}{\mathrm{Cos}\left(\frac{\pi}{8}\right)}\text{ ft} \approx 1.08\text{ ft.}$$



That's all for now.
Next issue I'll have solutions to K1, K2, L2, L3, and more.

Software Reviews

Edited by Shao Mah
Red Deer College
Red Deer, AB, Canada T4N 5H5

Readers are welcome to submit reviews of their favorite computer software to the editor for publication.

Title: EPIC—Exploration Programs in Calculus
Distributor: Prentice-Hall, Inc.
Cost: $29.50
Computer: IBM PC or compatible
Memory Needs: 64K
Peripherals: 1 disk drive, graphics card, monitor, preferably color
Description: This graphing package contains programs on straight lines, function plotting and evaluation, limits and derivatives, integration, transcendental functions, and polar and parametric equations. Functions are input with conventional symbols — no computer language is required. Graphing is done quickly, yet can be slowed for observation of direction in a polar or parametric equation. Once a function has been plotted, the x- and y-intervals may be changed easily, the maximum and minimum obtained, and the inverse function or first or second derivative graphed with just one keystroke. The use of parameters in the function plotting and the transcendental function programs provides a means of visualizing ordinate addition, reflections, magnifications and translations. The integration program computes the area between curves using left, right, or midpoint Riemann sums on subintervals from 1 to 256 in number. EPIC delivers a "big bang for the bucks."
Reviewer: Myrna Mitchell, Pima Community College, P.O. Box 5027, Tucson, AZ 85703

Mathematical Trivia (MT)

by Ralph Selig

TRIVIA ANSWERS:

MT 1. a. Beginners All Purpose Symbolic Instruction Code
b. Formula Translator
c. Common Business Oriented Language

MT 2. 7 x 7 = 49, 4 x 9 = 36, 3 x 6 = 18, 1 x 8 = 8

MT 3. Artificial Numbers

MT 4. Blaise Pascal 1642

MT 5. Oliver Wendell Holmes — "The One-Hoss Shay"

Questions appear throughout journal. Look for them.

AMATYC Reviewers

EDITORIAL PANEL

Agnes Azzolino	Middlesex County College	Edison, NJ
Ione Boodt	Univ. of Indianapolis	Indianapolis, IN
Mike Davidson	Modesto J.C.	Modesto, CA
Carla Thompson	Tulsa Junior College	Tulsa, OK
Brian Hickey	East Central College	Union, MO
Gordon W. Hoagland	Ricks College	Rexburg, ID
Nelson G. Rich	Nazareth College	Rochester, NY
Martha M. Wood	Clayton State College	Morrow, GA

REVIEWER PANEL

Darrell H. Abney	Maysville C.C.	Maysville, KY
Paul E. Bland	Eastern Kentucky University	Richmond, KY
Wilfred E. Boykin	Univ. Central Ct. State	New Britain, CT
Randall Brian	Vincennes University	Vincennes, IN
Pansy Brunson	Western Kentucky University	Bowling Green, KY
Alan Chutsky	Queensborough C.C.	Bayside, NY
Patricia Costello	Eastern Kentucky University	Richmond, KY
John DeCoursey	Vincennes University	Vincennes, IN
George Dorner	Harper College	Palatine, IL
David Dyer	Prince Georges	Largo, MD
Joseph R. Fieldler	Ohio State University	Columbus, OH
Gregory D. Foley	Ohio State University	Columbus, OH
Barbara R. Gab	Prince Georges	Largo, MD
Anne Grams	Embry Riddle Aeronautical Univ.	Daytona Beach, FL
Robert J. Greger	Harford C.C.	Bel Air, MD
Ronald J. Harkins	Miami University	Hamilton, OH
Peter Herron	Suffolk C.C.C.	Selden, NY
Bernard G. Hoerbelt	Genesee C.C.	Batavia, NY
Calvin J. Holt	Paul D. Camp C.C.	Franklin, VA
James T. Johnson	Modesto Jr. College	Modesto, CA
Deborah Levine	Nassau C.C.	Garden City, NY
Ved P. Madan	Red Deer College	Alberta, CANADA
Pamela E. Matthews	Willamette University	Salem, OR
Mary McCarty	Sullivan C.C.C.	Loch Sheldrake, NY
Terry A. Nyman	UW-Fox Valley	Menasha, WI
Carol Olmstead	Univ. of Nevada	Reno, NV
Loren W. Pixley	Richland C.C.	Decatur, IL
Dennis Reissig	Suffolk C.C.C.	Selden, NY
Jack W. Rotman	Lansing C.C.	Lansing, MI
Karen J. Schroeder	Bentley College	Waltham, MA
Ralph Selig	Fairleigh Dickinson University	Hakensack, NJ
Lowell Stultz	Kalamazoo Valley C.C.	Kalamazoo, MI
Leslie R. Tanner	The College of Idaho	Caldwell, ID
Travis Thompson	Harding University	Searcy, AR
August J. Zarcone	College of DuPage	Glen Ellyn, IL

WHY JOIN AMATYC?

The American Mathematical Association of Two-Year Colleges (AMATYC) was established in 1974 to provide a unique, national forum for two-year college mathematics educators. Today, AMATYC is the only national organization that exclusively serves the needs and purposes of this group.

AMATYC holds a national convention annually in a major city. AMATYC encourages two-year college mathematicians to assume responsible leadership positions; to interact on an equal basis with four-year university personnel; to be members on a proportional basis of national steering or policy committees; to represent at the national level the concerns of two-year college mathematics educators. *The AMATYC Review* provides an opportunity to publish articles by and for two-year college faculty. A national newsletter is published three times yearly for all members. AMATYC is a member of the National Conference Board of the Mathematical Sciences.

CUT HERE, MAIL TO TREASURER

INDIVIDUAL MEMBERSHIP APPLICATION The American Mathematical Association of Two-Year Colleges

MAIL TO: Karen Sharp, AMATYC Treasurer, Mott Community College, Flint, MI 48502

MEMBER'S NAME _____ POSITION _____

COLLEGE _____

COLLEGE ADDRESS _____ PHONE _____

CITY _____ STATE _____ ZIP _____

RESIDENCE ADDRESS _____ PHONE _____

Indicate preferred mailing address: ☐ College ☐ Residence

All payments in U.S. funds payable to AMATYC. Membership fee:

☐ $25 Yearly Individual Membership (any person interested in two-year college mathematics)
☐ $10 Yearly Associate Membership (full-time student)

Name of AMATYC Sponsor _____
☐ $22 Reduced Yearly Membership for members of organizations affiliated with AMATYC (see below)
☐ $500 Life Membership

Please check appropriate state affiliation:

☐ AlaMATYC (Alabama)
☐ ArizMATYC (Arizona)
☐ ARKMATYC (Arkansas)
☐ CMC³ (California)
☐ CMC³ South (California)
☐ MATYCONN (Connecticut)
☐ FTYCMA (Florida)
☐ IMACC (Illinois)
☐ IRMC (Indiana)
☐ KYMATYC (Kentucky)
☐ MMATYC (Maryland)
☐ MICHMATYC (Michigan)
☐ MOMATYC (Missouri)
☐ NEMATYC (New England)
☐ MATYCNJ (New Jersey)
☐ OHIOMATYC (Ohio)
☐ OJCMA (Oklahoma)
☐ PSMATYC (Pennsylvania)
☐ SOCAMATYC (South Carolina)
☐ TMATYC (Tennessee)
☐ TexMATYC (Texas)
☐ VMATYC (Virginia)
☐ WAMATYC (Washington)
☐ WYMATYC (Wyoming)

Membership includes *The AMATYC Review* and *The AMATYC News*. In addition, the following journals are available at extra cost:

☐ Mathematics and Computer Education Journal $16 for 3 issues
☐ The College Mathematics Journal $17 for 5 issues
☐ UMPA Journal $25 for 4 issues (U.S. subscriptions only)

Total Amount Enclosed _____ Date Paid _____

AMATYC Institutional Members

Addison-Wesley Publishing College	Reading, MA	01867
Brooks-Cole Publishing Co.	Monterey, CA	93940
Bucks County Comm. College	Newtown, PA	18940
Burlington County College	Pemberton, NJ	08068
Butte College	Oroville, CA	95965
C. S. Mott Community College	Flint, MI	48503
Casper College	Casper, WY	82601
Chabot College	Hayward, CA	94545
Charles Co. Community College	La Plata, MD	20646
Chemeketa Community College	Salem, OR	97309
Clackamas Community College	Oregon City, OR	97045
Clark Technical College	Springfield, OH	45501
Columbus Technical Institute	Columbus, OH	43215
D.C. Heath and Co.	Lexington, MA	02173
DeKalb Community College-South	Decatur, GA	30034
Delaware Tech. & Comm. College	Newark, DE	19702
Delta College	University Center, MI	48710
EN Boekhandel B V	1000 GC Amster., Hol., Netherland	
Florida College	Temple Terra, FL	33617
Florida Jr. College-Jacksonville	Jacksonville, FL	32205
Fullerton College	Fullerton, CA	92634
George Brown College	Toronto, Ontario, Canada	M5T 2T9
H&H Publishing Company, Inc.	Clearwater, FL	33575
Harford Community College	Bel Air, MD	21014
Harper College	Palatine, IL	60067
Houghton Mifflin Company	Boston, MA	02107
Hudson County Community College	Jersey City, NJ	07306
Idaho State University	Pocatello, ID	83201
Illinois State University	Normal, IL	61761
Johnson Co. Community College	Overland Park, KS	66201
John Wiley and Sons	New York, NY	10158
Lakeland College	Vermilion, Alberta, Canada	TOB 4MO
Lane Comm. College/Dept. of Math	Eugene, OR	97405
Matanuska-Susitna Community College	Palmer, AK	99645
Monroe Community College	Rochester, NY	14623
Moraine Valley Community College	Palos Hills, IL	60645
Montgomery College	Tacoma Park, MD	20912
New Mexico Military Inst.	Roswell, NM	88201
North Hennepin Comm. College	Minneapolis, MN	55455
North Idaho College	Cpeir D'Alene, ID	83814
Northern Arizona University	Flagstaff, AZ	86011
Penn State Univ.-Capitol Campus	Middleton, PA	17057
Pennsylvania State University	Dunmore, PA	18512
Pikes Peak Community College	Colorado Springs, CO	80906
Polk Community College	Winter Haven, FL	33881
Portland Community College	Portland, OR	97219
Prentice-Hall Inc.	Englewood Cliffs, NJ	07632
Raymond Walters College	Cincinnati, OH	45236
Richland Community College	Decatur, IL	62526
Ricks College	Rexburg, ID	83440
Saunders College Publishing Co.	Philadelphia, PA	19105
Schoolcraft College	Livonia, MI	48152
Scott Foresman Company	Glenview, IL	60025
Southwest Texas State University	San Marcos, TX	78666
Southwest Virginia Comm. College	Richlands, VA	24641
St. Bernard Parrish Comm. College	Chalmette, LA	70043
Stark Technical College	N. Canton, OH	44720
Tarrant Community College	Hurst, TX	76054
Three Rivers Community College	Poplar Bluff, MO	63901
University of Puerto Rico	Bayamon, PR	00620
Volunteer State Community College	Gallatin, TN	37066
Wallace State Community College	Selma, AL	36702
Waterbury State Technical College	Waterbury, CT	06708
West Publishing Company	Moss Beach, CA	94038
Westark Community College	Ft. Smith, AR	72913

INSTITUTIONAL MEMBERSHIP

Colleges, publishers, and other institutions can support AMATYC by being Institutional Members. As indicated below, the annual membership fee is $75. Subscriptions to *The AMATYC Review* and *The AMATYC News* are included in this fee. Of particular importance to collegiate institutional members will be the new AMATYC committee for mathematics department chairpersons.

An additional benefit of an institutional membership is one *complimentary* AMATYC convention registration. Future conventions, which are held in the Fall, will be in Calgary (1988) and Baltimore (1989). Institutional members support the excellence of the programs at these annual conventions.

INSTITUTIONAL MEMBERSHIP APPLICATION

The American Mathematical Association of Two-Year Colleges

Mail to:
Karen Sharp, AMATYC Treasurer, Mott Community College, Flint, Michigan 48502

AMATYC COLLEGE CONTACT PERSON _____ POSITION _____

NAME OF INSTITUTION _____

ADDRESS _____

CITY _____ STATE _____ ZIP _____

Membership Fee — $75 in U.S. funds payable to AMATYC (includes *The AMATYC Review*, *The AMATYC News*, and one complimentary convention registration)

American Mathematical Association of Two-Year Colleges

EXECUTIVE BOARD OFFICERS
1987-1989

President
Karl J. Smith
Santa Rosa Junior College
Santa Rosa, CA 95401
(707) 527-4356

President-Elect
Dale Ewen
Parkland Community College
2400 West Bradley Avenue
Champaign, IL 61821
(217) 351-2225

Treasurer
Karen Sharp
Mott Community College
Flint, MI 48502
(313) 762-0279/0277

Secretary
Cheryl Cleaves
State Technical Institute
Memphis, TN 38134
(901) 377-4110

Past President
Stephen B. Rodi
Austin Community College
P.O. Box 2285
Austin, TX 78768
(512) 495-7222

REGIONAL VICE PRESIDENTS

Northeast
Allen R. Angel
Monroe Community College
Rochester, NY 14623
(716) 424-5400 Ext. 5215

Mid-Atlantic
James Newsom
Tidewater Comm. College
1700 Coll Crescent
Virginia Beach, VA 23456
(703) 427-7100

Southwest
Marilyn Mays
1122 Woodmount CT.
Denton, TX
(817) 387-0764

Southeast
Margie Hobbs
State Technical Institute
Memphis, TN 38134
(901) 377-4110

Central
Ron Beeler
East Central Jr. College
Box 529
Union, MO 63084
(314) 583-5193

Northwest
Barbara Poole
North Seattle Community College
Seattle, WA 98103
(206) 527-3600

Midwest
James Hall
Parkland Community College
2400 West Bradley Avenue
Champaign, IL 61821
(217) 351-2225

West
Wanda Garner
Cabrillo College
6500 Sequel Dr.
Aptos, CA 95003
(408) 688-2075

ACADEMIC COMMITTEES

Developmental Mathematics
Bob Maleda
Allegheny County CC South
West Mifflin, PA 15122

Academic Computer
Wade Ellis
West Valley College
Saratoga, CA 95070

Student Mathematics League
Steve Blasberg
West Valley College
Saratoga, CA 95070

Equal Opportunity
Jean Smith
Middlesex Community College
Middletown, CT 06457

Technical Mathematics
Ray Collings
Tri County Technical College
Pendleton, SC 29670

Education
Geoffrey Akst
Borough of Manhattan C.C.
199 Chambers St.
New York, NY 10007

ERIC Clearinghouse for
Junior Colleges OCT 14 1988